BOUNDS ON ORTHONORMAL POLYNOMIALS FOR RESTRICTED MEASURES

D. S. LUBINSKY

Abstract. Suppose that ν is a given positive measure on $[-1,1]$, and that μ is another measure on the real line, whose restriction to $(-1,1)$ is ν. We show that one can bound the orthonormal polynomials $p_n(\mu, y)$ for μ and $y \in \mathbb{R}$, by the supremum of $\left|S_J(y)\, p_{n-J}\left(S_J^2 \nu, y\right)\right|$, where the sup is taken over all $0 \leq J \leq n$ and all monic polynomials S_J of degree J with zeros in an appropriate set.

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orthogonal polynomials, bounds

1. Results

Let μ be a finite positive Borel measure on the real line with infinitely many points in its support, and all finite moments

$$\int t^j d\mu(t),\ j = 0, 1, 2, \ldots .$$

Then we may define orthonormal polynomials

$$p_n(\mu, x) = \gamma_n(\mu)\, x^n + \ldots, \gamma_n(\mu) > 0,$$

$n = 0, 1, 2, \ldots$ satisfying the orthonormality conditions

$$\int p_n(\mu, x)\, p_m(\mu, x)\, d\mu(x) = \delta_{mn}.$$

The nth reproducing kernel is

$$K_n(\mu, x, t) = \sum_{j=0}^{n-1} p_j(\mu, x)\, p_j(\mu, t).$$

For real polynomials S, $p_n\left(S^2 \nu, x\right)$ denotes the nth orthonormal polynomial for the measure $S^2 \nu$.

A central problem in the theory of orthonormal polynomials is to establish bounds on $p_n(\mu, x)$. Indeed, one of the most celebrated questions in this subject is the Steklov Conjecture, posed in 1920 and solved by E. A. Rakhmanov in 1979, 1981 [27], [28]. Steklov conjectured that if μ has support $[-1,1]$, and is absolutely continuous there, while μ' is bounded below by a positive constant, then $\{p_n(\mu, x)\}_{n \geq 0}$ is uniformly bounded in $[-1,1]$, that is

$$\sup_{n \geq 0} \sup_{x \in [-1,1]} |p_n(\mu, x)| < \infty.$$

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SORRY, THIS IS OF COURSE WRONG AND IS WRONG IN THE PUBLISHED VERSION. IT SHOULD BE THAT FOR EACH $0 < \rho < 1$,

$$\sup_{n \geq 0} \sup_{x \in [-\rho,\rho]} |p_n(\mu, x)| < \infty.$$

Rakhmanov showed that Steklov's conjecture is false, and in fact one can find a measure satisfying its hypotheses such that for infinitely many n,

$$|p_n(\mu, 0)| \geq C n^{1/2} / (\log n)^{3/2+\varepsilon}, \tag{1.1}$$

any $\varepsilon > 0$.

Rakhmanov's weight μ' was unbounded. Ambroladze in 1989 [3] constructed a positive continuous weight on the unit circle for which the orthonormal polynomials are unbounded, and in 1991 [4] a continuous weight such that for infinitely many n, their value at 0 grows almost as fast as $\log n$. He even constructed weights on the unit circle that satisfy a weak Dini condition, that is for some $\varepsilon \in (0, 1)$,

$$|\mu'(s) - \mu'(t)| \leq L |\ln|s-t||^{-(1-\varepsilon)}, \; s, t \in [0, 2\pi]$$

but the corresponding sequence of orthonormal polynomials can be unbounded.

There have been further major developments in recent years. Aptekarev, Denisov and Tulyakov [5], [6] showed that one can remove the factor $(\log n)^{3/2+\varepsilon}$ in (1.1). They proved that if we consider all orthonormal polynomials arising from a measure whose derivative is bounded below by a given positive constant on the circle, then the supremum of the sup norm over the unit circle of the nth orthonormal polynomial over all such measures grows like $\sqrt{n}$. In subsequent work, Denisov, Aptekarev, and Rush have explored the possible rates of growth when there are additional conditions, such as both μ' and $1/\mu'$ being bounded, or belonging to BMO [1], [11], [12], [13], [14], [15].

What about positive results that establish boundedness? Probably the most general result for measures supported on $[-1, 1]$ is still an old 1979 result of Badkov [8]. He proved that if μ satisfies Szegő's condition

$$\int_{-1}^{1} \frac{\log \mu'(x)}{\sqrt{1-x^2}} dx > -\infty$$

and μ' satisfies a local Dini-Lipschitz condition, that is in some interval $[a, b] \subset (-1, 1)$, the modulus of continuity

$$\omega_{[a,b]}(\delta) = \sup \{|\mu'(s) - \mu'(t)| : s, t \in [a, b] \text{ and } |s-t| \leq \delta\}, \delta > 0 \tag{1.2}$$

satisfies

$$\int_0^1 \frac{\omega_{[a,b]}(t)}{t} dt < \infty, \tag{1.3}$$

then the corresponding orthonormal polynomials are uniformly bounded in compact subsets of (a, b). This was a consequence of deeper pointwise asymptotics.

For non-Szegő weights, Rakhmanov showed in 1986 [29] that for weights on the unit circle that satisfy a Dini-Lipschitz condition on the whole circle, the orthonormal polynomials are uniformly bounded below on arcs where the weight is bounded below by a positive number. The relationship between spacing of zeros of orthogonal polynomials and bounds has been explored in [22], [24].

For exponential weights, supported on finite or infinite intervals, Eli Levin and the author [20] established global bounds, using extensions of a technique that goes back to Stan Bonan [9], a student of Paul Nevai. Extensions of that method due to Mhaskar appear in [25] [26]. For varying exponential weights, the method was applied in [21]. In some of these results, bounds are then superseded by asymptotics.

There is one classic technique, called Korous's method, that allows one to transfer bounds from one set of orthonormal polynomials $\{p_n(\mu,x)\}$ to another $\{p_n(\nu,x)\}$, provided ν'/μ' satisfies a smoothness condition - either a Dini condition or a Lipschitz 1/2 type condition, depending on what one knows about $\{p_n(\mu,x)\}$. It is a powerful and widely used technique, but requires starting with knowledge about a base set of polynomials $\{p_n(\mu,x)\}$ [16, Section 1.7].

What happens if we know that our measure μ has a given restriction ν when restricted to $(-1,1)$? Can we bound the orthonormal polynomial $p_n(\mu,y)$ in some way in terms of orthogonal polynomials related to $p_n(\nu,y)$? Indeed one may, as shown by the following. It involves $p_n\left(S^2\nu,y\right)$, the orthonormal polynomial for the measure $S^2\nu$, where S is a real polynomial.

Theorem 1.1
Let ν be a positive measure on $[-1,1]$, with infinitely many points in its support. Let $\mathcal{K}$ be a closed subset of $\mathbb{R}$ containing $(-1,1)$. Let $y\in\mathbb{R}$ and $n\geq 1$. Then

$$\begin{aligned}&\sup\left\{p_n^2(\mu,y):\mu_{|(-1,1)}=\nu\text{ and }\operatorname{supp}[\mu]\subseteq\mathcal{K}\right\}\\=&\sup S_J^2(y)\,p_{n-J}^2\left(S_J^2\nu,y\right),\end{aligned}$$

where the supremum is taken over all $0\leq J\leq n$ and monic polynomials S_J of degree J with distinct zeros in $\mathcal{K}\setminus(-1,1)$.

2025 REMARK
There is a small inaccuracy here and the same problem appears in the published version. In the proofs we assume $\mu\geq\nu$. This is automatic in the main and most common case when ν does not have masspoints at ±1, but might not be true when ν has masspoints at ±1. So it should really be,

$$\sup\left\{p_n^2(\mu,y):\mu_{|(-1,1)}=\nu,\mu\geq\nu,\text{ and }\operatorname{supp}[\mu]\subseteq\mathcal{K}\right\}$$

Remarks
(a) Note that we take the restriction of μ to $(-1,1)$ and not $[-1,1]$. This allows the measure μ to add masspoints at ±1 to ν.
(b) If $J=0$, we take $S_J=1$.
(c) The set of polynomials $\{S_J\}$ does not depend on the particular y, nor on the measure μ, but only on the set $\mathcal{K}$.
(d) It is a non-trivial task to turn the qualitative formula above into quantitative estimates. This can be done using potential theory for external fields Q, where $S_J^2=e^{-2Q}$. We shall do this elsewhere.
(e) It is not clear if the sup above is attained in general. It may be bounded above by $K_{n+1}(\nu,y,y)$ because of monotonicity of Christoffel functions and the fact that $\mu\geq\nu$. Indeed,

$$p_n^2(\mu,y)\leq K_{n+1}(\mu,y,y)\leq K_{n+1}(\nu,y,y).$$

This is not a good bound, but is indicative that the supremum above can grow like a power of n in the worst cases. Suppose, as an example, that ν is the Legendre measure so that $\nu' = 1$, and that $\mathcal{K} = [-2, 2]$. Then considering $J = n$, we see that for even n, we may effectively take (as a limiting case) $S_n(t) = \left(1 - t^2\right)^{n/2}$ and

$$S_n^2(0)\, p_0^2\left(S_n^2 \nu, 0\right) = 1 / \int_{-1}^{1} \left(1 - t^2\right)^n dt = \sqrt{\frac{n}{\pi}}\,(1 + o(1)).$$

(f) Denisov [12] investigated the size of orthogonal polynomials for measures on the unit circle when one adds point masses to Lebesgue measure, obtaining quantitative estimates, but focusing on the case where the mass points are inside the support of the original measure.

When estimating $p_n(\mu, y)$ for a fixed μ, it might be better to restrict the zeros of S_J. One choice is to use a sequence of sets of points outside $(-1, 1)$, which are "asymptotically dense". Thus for $L \geq 1$, let $\mathcal{B}_L$ denote a set of L points in $\mathbb{R} \backslash (-1, 1)$, such that for all $t \in \mathcal{K}$,

$$\lim_{L \to \infty} dist\,(t, \mathcal{B}_L) = 0. \tag{1.4}$$

Here dist denotes the usual distance from a point to a set of points.

Another option is to use Gauss quadrature formulae of order $n+1$ for $(\mu - \nu)_{|1,\infty)}$ and $(\mu - \nu)_{|(-\infty,-1]}$. More precisely, we use the points $\left\{b_j^+\right\}_{j=1}^{n+1}$ in the Gauss quadrature

$$\int_1^{\infty} P(t)\, d(\mu - \nu)(t) = \sum_{j=1}^{n+1} \lambda_j^+ P\left(b_j^+\right),$$

valid for polynomials P of degree $\leq 2n+1$. Observe that if ν has a mass point at 1, that mass point is excluded from the restriction of $\mu - \nu$ to $[1, \infty)$, but additional mass points of μ are included. Similarly for $(-\infty, -1]$.

Theorem 1.2

Let ν be a positive measure on $[-1, 1]$, with infinitely many points in its support. Let $\mathcal{K}$ be a closed subset of $\mathbb{R}$ containing $(-1, 1)$. Let μ be a measure with support in $\mathcal{K}$ and with all finite moments, such that $\mu_{|(-1,1)} = \nu$. Let $y \in \mathbb{R}$ and $n \geq 1$.
(a) For $L \geq 1$, let $\mathcal{B}_L$ denote a set of L points in $\mathbb{R} \backslash (-1, 1)$, such that for all $t \in \mathcal{K}$, (1.4) holds. Let M_L denote

$$\sup S_J^2(y)\, p_{n-J}^2\left(S_J^2 \nu, y\right),$$

where the supremum is taken over all $0 \leq J \leq n$ and monic polynomials S_J of degree J with distinct zeros in $\mathcal{B}_L$. Then

$$p_n^2(\mu, y) \leq \limsup_{L \to \infty} M_L.$$

(b) If both $(\mu - \nu)_{|1,\infty)}$ and $(\mu - \nu)_{|(-\infty,-1]}$ have at least $n+1$ points in their supports, let $\mathcal{B}_G$ denote the set of $2n+2$ points formed from the union of the Gauss quadrature points of order $n+1$ for these measures. If one or both of them has less than $n+1$ points in its support, we instead use the mass points from that or those measures in forming $\mathcal{B}_G$. Then

$$p_n^2(\mu, y) \leq \sup S_J^2(y)\, p_{n-J}^2\left(S_J^2 \nu, y\right),$$

where the supremum is taken over all $0 \leq J \leq n$ *and monic polynomials* S_J *of degree* J *with distinct zeros in* $\mathcal{B}_G$.

The idea of proof is first to consider measures of the form

$$\mu = \nu + \sum_{j=1}^{L} \alpha_j \delta_{b_j}, \tag{1.5}$$

and to show that if all $\alpha_j \in [T, S]$, then $|p_n(\mu, y)|$ is largest when each $\alpha_j = S$ or T. That is, the maximum is attained at the vertices of $[T, S] \times [T, S] \times ... \times [T, S]$.

Theorem 1.3
Let ν *be a positive measure on* $[-1, 1]$, *with infinitely many points in its support. Let* $L \geq 1$, $\{b_j\}_{j=1}^{L}$ *be distinct real numbers,* $y \in \mathbb{R}$ *and* $n \geq 1$.
(a) Let $S > T \geq 0$. *Let*

$$\Omega^*(S, T) = \sup p_n^2(\mu, y)$$

where the sup is taken over all measures μ *of the form (1.5) with all* $\alpha_j \in [T, S]$. *Then* $\Omega^*(S, T)$ *is attained for some measure of the form (1.5) where all* $\alpha_j \in \{S, T\}$. *If also* $p_n(\nu, y) \neq 0$, *then* $\Omega^*(S, T)$ *is attained only for such measures unless* $p_n(\mu, y)$ *is a constant function of some* α_j, *that is for some* $1 \leq j \leq L$, $p_n(\mu, y)$ *is independent of the choice of* α_j .
(b) Let

$$\Omega^* = \sup p_n^2(\mu, y)$$

where the sup is taken over all measures μ *of the form (1.5) with all* $\alpha_j \geq 0$. *Then*

$$\Omega^* = \sup S_J^2(y)\, p_{n-J}^2\left(S_J^2 \nu, y\right)$$

where the sup is taken over all $0 \leq J \leq \min\{L, n\}$ *and all monic polynomials* S_J *of degree* J *with distinct zeros in* $\{b_j\}_{j=1}^{L}$.

This paper is organized as follows: we relate $p_n(\mu, y)$ to $p_n(\nu, y)$ using standard tools in Section 2. We also show that in the case $\alpha_j \to \infty$, we are forcing $p_n(\mu, t)$ to have a factor $t - b_j$. In Section 3, we show that $p_n^2(\mu, y)$ is largest either when $\alpha_j = T$ or $\alpha_j = S$ or when the value is independent of α_j. In Section 4, we prove Theorems 1.1 to 1.3. In the sequel $C, C_1, C_2, ...$ denote constants independent of n, x, t. The same symbol does not necessarily denote the same constant in different occurences.

2. Adding a Discrete Measure

There is a very extensive literature on adding masspoints to an existing measure. See for example, results and references in [2], [7], [10], [12], [17], [18], [30], [31]. In this section, we use standard techniques in this topic. We are not sure if any of the formulae we establish are new. Let $L \geq 1$ and

$$\mu = \nu + \sum_{j=1}^{L} \alpha_j \delta_{b_j}, \tag{2.1}$$

where all $\alpha_j > 0$, and all b_j are distinct and real. Let

$$\mathbf{D} = \begin{bmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \dots & \alpha_L \end{bmatrix}; \mathbf{K}_n = \left[K_n\left(\nu, b_j, b_k\right)\right]_{1\le j,k\le L}; \tag{2.2}$$

and

$$\mathbf{C}_n = \mathbf{D}^{-1} + \mathbf{K}_n. \tag{2.3}$$

Note that $\mathbf{D}$ is positive definite, while $\mathbf{K}_n$ is symmetric and positive semidefinite. The latter follows from the identity

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_L \end{bmatrix}^T \mathbf{K}_n \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_L \end{bmatrix}^T = \sum_{i=0}^{n-1} \left(\sum_{j=1}^{L} u_j p_i\left(b_j\right) \right)^2.$$

It follows that $\mathbf{C}_n$ is also positive definite. Let us fix a y with

$$p_n\left(\nu, y\right) \neq 0 \tag{2.4}$$

and set

$$X = \begin{bmatrix} p_n\left(\nu, b_1\right) \\ p_n\left(\nu, b_2\right) \\ \vdots \\ p_n\left(\nu, b_L\right) \end{bmatrix}; Y = -\frac{1}{p_n\left(\nu, y\right)} \begin{bmatrix} K_n\left(\nu, y, b_1\right) \\ K_n\left(\nu, y, b_2\right) \\ \vdots \\ K_n\left(\nu, y, b_L\right) \end{bmatrix}.$$

(2.5)

Theorem 2.1
With the above hypotheses,

$$\Gamma = \left(\frac{p_n\left(\mu, y\right)}{p_n\left(\nu, y\right)} \right)^2 = \frac{\left\{1 + X^T \mathbf{C}_n^{-1} Y\right\}^2}{1 + X^T \mathbf{C}_n^{-1} X}. \tag{2.6}$$

Also,

$$\left(\frac{\gamma_n\left(\mu\right)}{\gamma_n\left(\nu\right)} \right)^2 = \frac{1}{1 + X^T \mathbf{C}_n^{-1} X}. \tag{2.7}$$

Proof
Let

$$\pi_{n-1}\left(s\right) = p_n\left(\mu, s\right) - \frac{\gamma_n\left(\mu\right)}{\gamma_n\left(\nu\right)} p_n\left(\nu, s\right).$$

Then using orthogonality, we see that

$$\begin{aligned}\pi_{n-1}(s) &= \int_{-1}^{1} K_n(\nu, s, t)\,\pi_{n-1}(t)\,d\nu(t) \\ &= \int_{-1}^{1} K_n(\nu, s, t)\,p_n(\mu, t)\,d\nu(t) \\ &= -\sum_{j=1}^{L} \alpha_j K_n(\nu, s, b_j)\,p_n(\mu, b_j)\,. \end{aligned} \tag{2.8}$$

Setting $s = b_k$,

$$p_n(\mu, b_k)\{1 + \alpha_k K_n(\nu, b_k, b_k)\} + \sum_{j=1, j\neq k}^{L} \alpha_j K_n(\nu, b_k, b_j)\,p_n(\mu, b_j) = \frac{\gamma_n(\mu)}{\gamma_n(\nu)} p_n(\nu, b_k)\,. \tag{2.9}$$

Let

$$\mathbf{B} = \begin{bmatrix} 1+\alpha_1 K_n(\nu, b_1, b_1) & \alpha_2 K_n(\nu, b_1, b_2) & \alpha_3 K_n(\nu, b_1, b_3) & \dots & \alpha_L K_n(\nu, b_1, b_L) \\ \alpha_1 K_n(\nu, b_2, b_1) & 1+\alpha_2 K_n(\nu, b_2, b_2) & \alpha_3 K_n(\nu, b_2, b_3) & \dots & \alpha_L K_n(\nu, b_2, b_L) \\ \alpha_1 K_n(\nu, b_3, b_1) & \alpha_2 K_n(\nu, b_3, b_2) & 1+\alpha_3 K_n(\nu, b_3, b_3) & \dots & \alpha_L K_n(\nu, b_3, b_L) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_1 K_n(\nu, b_L, b_1) & \alpha_2 K_n(\nu, b_L, b_2) & \alpha_3 K_n(\nu, b_L, b_3) & \dots & 1+\alpha_L K_n(\nu, b_L, b_L) \end{bmatrix}$$

$$= \mathbf{I} + \mathbf{K}_n\mathbf{D} = \mathbf{C}_n\mathbf{D}.$$

Note that then $\mathbf{B}$ is non-singular as $\mathbf{C}_n, \mathbf{D}$ are, and

$$\mathbf{D}\mathbf{B}^{-1} = \mathbf{C}_n^{-1}. \tag{2.10}$$

We can recast (2.9) as

$$\mathbf{B}\begin{bmatrix} p_n(\mu, b_1) \\ p_n(\mu, b_2) \\ \vdots \\ p_n(\mu, b_L) \end{bmatrix} = \frac{\gamma_n(\mu)}{\gamma_n(\nu)} \begin{bmatrix} p_n(\nu, b_1) \\ p_n(\nu, b_2) \\ \vdots \\ p_n(\nu, b_L) \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} p_n(\mu, b_1) \\ p_n(\mu, b_2) \\ \vdots \\ p_n(\mu, b_L) \end{bmatrix} = \frac{\gamma_n(\mu)}{\gamma_n(\nu)}\mathbf{B}^{-1} \begin{bmatrix} p_n(\nu, b_1) \\ p_n(\nu, b_2) \\ \vdots \\ p_n(\nu, b_L) \end{bmatrix}. \tag{2.11}$$

Then from (2.8), and this last relation,

$$\pi_{n-1}(y) = -\frac{\gamma_n(\mu)}{\gamma_n(\nu)} \begin{bmatrix} \alpha_1 K_n(\nu, y, b_1) \\ \alpha_2 K_n(\nu, y, b_2) \\ \vdots \\ \alpha_L K_n(\nu, y, b_L) \end{bmatrix}^T \mathbf{B}^{-1} \begin{bmatrix} p_n(\nu, b_1) \\ p_n(\nu, b_2) \\ \vdots \\ p_n(\nu, b_L) \end{bmatrix}.$$

$$\Rightarrow p_n(\mu, y) = \frac{\gamma_n(\mu)}{\gamma_n(\nu)} \left\{ p_n(\nu, y) - \begin{bmatrix} \alpha_1 K_n(\nu, y, b_1) \\ \alpha_2 K_n(\nu, y, b_2) \\ \vdots \\ \alpha_L K_n(\nu, y, b_L) \end{bmatrix}^T \mathbf{B}^{-1} \begin{bmatrix} p_n(\nu, b_1) \\ p_n(\nu, b_2) \\ \vdots \\ p_n(\nu, b_L) \end{bmatrix} \right\}.$$

Then

$$\begin{aligned} \frac{p_n(\mu, y)}{p_n(\nu, y)} &= \frac{\gamma_n(\mu)}{\gamma_n(\nu)} \left\{ 1 + Y^T \mathbf{D} \mathbf{B}^{-1} \begin{bmatrix} p_n(\nu, b_1) \\ p_n(\nu, b_2) \\ \vdots \\ p_n(\nu, b_L) \end{bmatrix} \right\} \\ &= \frac{\gamma_n(\mu)}{\gamma_n(\nu)} \left\{ 1 + Y^T \mathbf{C}_n^{-1} X \right\}, \end{aligned}$$

(2.12)

by (2.10). This will give (2.6) after we prove (2.7). To prove (2.7), we evaluate $\int_{-1}^{1} \pi_{n-1}^2 d\nu$ in two different ways. First, from the definition of π_{n-1},

$$\begin{aligned} & \int_{-1}^{1} \pi_{n-1}^2(s)\, d\nu(s) \\ = & \int_{-1}^{1} p_n^2(\mu, s)^2\, d\nu(s) - 2\left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2 + \left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2 \\ = & \ 1 - \sum_{j=1}^{L} \alpha_j p_n(\mu, b_j)^2 - \left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2. \end{aligned}$$

Also, from (2.8),

$$\begin{aligned} & \int_{-1}^{1} \pi_{n-1}^2(s)\, d\nu(s) \\ = & \int_{-1}^{1} \left(-\sum_{j=1}^{L} \alpha_j K_n(\nu, s, b_j)\, p_n(\mu, b_j) \right)^2 d\nu(s) \\ = & \sum_{j,k=1}^{L} \alpha_j \alpha_k K_n(\nu, b_j, b_k)\, p_n(\mu, b_j)\, p_n(\mu, b_k). \end{aligned}$$

Combining the last two identities, and then using (2.11),

$$\begin{aligned} & 1-\left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2 \\ = & \sum_{j,k=1}^{L} \alpha_j\alpha_k K_n(\nu,b_j,b_k)\,p_n(\mu,b_j)\,p_n(\mu,b_k)+\sum_{j=1}^{L}\alpha_j p_n(\mu,b_j)^2 \\ = & \begin{bmatrix} p_n(\mu,b_1) \\ p_n(\mu,b_2) \\ \vdots \\ p_n(\mu,b_L)\end{bmatrix}^T (\mathbf{D}\mathbf{K}_n\mathbf{D}+\mathbf{D}) \begin{bmatrix} p_n(\mu,b_1) \\ p_n(\mu,b_2) \\ \vdots \\ p_n(\mu,b_L)\end{bmatrix} \\ = & \left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2 \left(\mathbf{B}^{-1}\begin{bmatrix} p_n(\nu,b_1) \\ p_n(\nu,b_2) \\ \vdots \\ p_n(\nu,b_L)\end{bmatrix}\right)^T (\mathbf{D}\mathbf{K}_n\mathbf{D}+\mathbf{D})\,\mathbf{B}^{-1}\begin{bmatrix} p_n(\nu,b_1) \\ p_n(\nu,b_2) \\ \vdots \\ p_n(\nu,b_L)\end{bmatrix} \\ = & \left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2 \begin{bmatrix} p_n(\nu,b_1) \\ p_n(\nu,b_2) \\ \vdots \\ p_n(\nu,b_L)\end{bmatrix}^T \left(\mathbf{B}^{-1}\right)^T \mathbf{D}\mathbf{B}\mathbf{B}^{-1} \begin{bmatrix} p_n(\nu,b_1) \\ p_n(\nu,b_2) \\ \vdots \\ p_n(\nu,b_L)\end{bmatrix}. \end{aligned}$$

Then

$$\left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2 \left\{1+\begin{bmatrix} p_n(\nu,b_1) \\ p_n(\nu,b_2) \\ \vdots \\ p_n(\nu,b_L)\end{bmatrix}^T \left(\mathbf{B}^{-1}\right)^T \mathbf{D} \begin{bmatrix} p_n(\nu,b_1) \\ p_n(\nu,b_2) \\ \vdots \\ p_n(\nu,b_L)\end{bmatrix}\right\}=1.$$

Here as $\mathbf{D},\mathbf{K}_n,\mathbf{C}_n$ are symmetric, from (2.10),

$$\mathbf{C}_n^{-1}=\left(\mathbf{C}_n^{-1}\right)^T=(\mathbf{D}\mathbf{B}^{-1})^T=\left(\mathbf{B}^{-1}\right)^T\mathbf{D}.$$

So

$$\left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2\left\{1+X^T\mathbf{C}_n^{-1}X\right\}=1.$$

Thus we have (2.7). Squaring and substituting in (2.12), gives (2.6). ■

We shall need another representation for Γ. We use a well known formula for determinants of block matrices involving Schur complements [19, p. 46, Ex. 15] . If $\mathbf{E},\mathbf{H}$ are square matrices and $\mathbf{H}$ is non-singular, while $\mathbf{F},\mathbf{G}$ have appropriate sizes:

$$\det\begin{bmatrix}\mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H}\end{bmatrix}=\det(\mathbf{E}-\mathbf{F}\mathbf{H}^{-1}\mathbf{G})\det(\mathbf{H}) \tag{2.13}$$

This follows from the identity

$$\begin{bmatrix}\mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H}\end{bmatrix}=\begin{bmatrix}\mathbf{I} & \mathbf{F}\mathbf{H}^{-1} \\ \mathbf{0} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\mathbf{E}-\mathbf{F}\mathbf{H}^{-1}\mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{0} \\ \mathbf{H}^{-1}\mathbf{G} & \mathbf{I}\end{bmatrix}.$$

.

Theorem 2.2
(a)

$$\left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2 = \frac{\det \mathbf{C}_n}{\det \mathbf{C}_{n+1}}. \tag{2.14}$$

(b)

$$\Gamma = \left(\frac{p_n(\mu, y)}{p_n(\nu, y)}\right)^2 = \frac{\left(\det\left(\mathbf{C}_n + XY^T\right)\right)^2}{\det \mathbf{C}_n \det \mathbf{C}_{n+1}}. \tag{2.15}$$

(c) $\frac{\det \mathbf{C}_n}{\det \mathbf{C}_{n+1}}$ is a decreasing function of each α_j.

Proof
(a) Using (2.13),

$$\det \begin{bmatrix} -1 & X^T \\ X & \mathbf{C}_n \end{bmatrix} = \left(-1 - X^T \mathbf{C}_n^{-1} X\right) \det \mathbf{C}_n.$$

We add $p_n(b_j) \times$ the first row (i.e. $\left[-1\ X^T\right]$) to the $(j+1)$st row in the left-hand side, for $j = 1, 2, ..., L$. Since $\mathbf{C}_n = \mathbf{D}^{-1} + \mathbf{K}_n$, we obtain (recall (2.2), (2.3), (2.5))

$$\det \begin{bmatrix} -1 & X^T \\ 0 & \mathbf{C}_{n+1} \end{bmatrix} = \left(-1 - X^T \mathbf{C}_n^{-1} X\right) \det \mathbf{C}_n.$$

Expanding the determinant in the left-hand side by the first column gives

$$(-1) \det \mathbf{C}_{n+1} = \left(-1 - X^T \mathbf{C}_n^{-1} X\right) \det \mathbf{C}_n.$$

So we obtain from (2.7),

$$\left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2 = \frac{1}{1 + X^T \mathbf{C}_n^{-1} X} = \frac{\det \mathbf{C}_n}{\det \mathbf{C}_{n+1}}.$$

(b) Similarly (2.13) gives

$$\det \begin{bmatrix} -1 & Y^T \\ X & \mathbf{C}_n \end{bmatrix} = \left(-1 - Y^T \mathbf{C}_n^{-1} X\right) \det \mathbf{C}_n. \tag{2.16}$$

Again, we add $p_n(\nu, b_j) \times$ the first row (i.e. $\left[-1\ Y^T\right]$) to the the $(j+1)$st row in the left-hand side, for $j = 1, 2, ..., L$. We obtain

$$\det \begin{bmatrix} -1 & Y^T \\ 0 & \mathbf{C}_n + XY^T \end{bmatrix} = \left(-1 - Y^T \mathbf{C}_n^{-1} X\right) \det \mathbf{C}_n.$$

Expanding by the first row and cancelling -1, gives

$$\det\left(\mathbf{C}_n + XY^T\right) = \left(1 + Y \mathbf{C}_n^{-1} X^T\right) \det \mathbf{C}_n.$$

Then from (2.6), (2.7), and (2.14),

$$\begin{aligned} \Gamma &= \left(\frac{p_n(\mu, y)}{p_n(\nu, y)}\right)^2 = \frac{\left\{1 + X^T \mathbf{C}_n^{-1} Y\right\}^2}{1 + X^T \mathbf{C}_n^{-1} X} \\ &= \left(\frac{\det\left(\mathbf{C}_n + XY^T\right)}{\det \mathbf{C}_n}\right)^2 \frac{\det \mathbf{C}_n}{\det \mathbf{C}_{n+1}}. \end{aligned}$$

(c) Since $\gamma_n(\mu)$ decreases as we increase μ, so $\frac{\det \mathbf{C}_n}{\det \mathbf{C}_{n+1}}$ decreases (monotonically, not necessarily strictly) as we increase any α_j. ■

Next, we see what happens when some $\alpha_j \to \infty$ and others approach 0 :

Lemma 2.3
Let $M > L \geq 1$. For $m \geq 1$, let

$$\mu_m = \nu + \sum_{j=1}^{L} \alpha_{m,j} \delta_{b_j} + \sum_{j=L+1}^{M} \alpha_{m,j} \delta_{b_j},$$

where for $1 \leq j \leq L$,

$$\lim_{m \to \infty} \alpha_{m,j} = \infty,$$

and for $L < j \leq M$,

$$\lim_{m \to \infty} \alpha_{m,j} = 0.$$

Let

$$S_L(t) = \prod\nolimits_{j=1}^{L} (t - b_j).$$

(a) Assume that $L \leq n$. Then uniformly for t in compact subsets of $\mathbb{C}$,

$$\lim_{m \to \infty} p_n(\mu_m, t) = S_L(t)\, p_{n-L}\left(S_L^2 \nu, t\right).$$

Also,

$$\lim_{m \to \infty} \gamma_n(\mu_m) = \gamma_{n-L}\left(S_L^2 \nu\right).$$

(b) Assume $L > n$. Then uniformly for t in compact subsets of $\mathbb{C}$,

$$\lim_{m \to \infty} p_n(\mu_m, t) = 0.$$

Proof
(a) We use the extremal property of the leading coefficients. First, as S_L has zeros at the first L masspoints $\{b_j\}$ of μ_m,

$$\begin{aligned}
\gamma_n(\mu_m)^{-2} &= \inf_{\substack{\deg(P)=n \\ P \text{ monic}}} \int P^2 d\mu_m \\
&\leq \int \left(\frac{S_L(t)\, p_{n-L}\left(S_L^2 \nu, t\right)}{\gamma_{n-L}\left(S_L^2 \nu\right)} \right)^2 d\nu(t) + \sum_{j=L+1}^{M} \alpha_{m,j} \left(\frac{S_L(b_j)\, p_{n-L}\left(S_L^2 \nu, b_j\right)}{\gamma_{n-L}\left(S_L^2 \nu\right)} \right)^2 \\
&= \gamma_{n-L}\left(S_L^2 \nu\right)^{-2} + \sum_{j=L+1}^{M} \alpha_{m,j} \left(\frac{S_L(b_j)\, p_{n-L}\left(S_L^2 \nu, b_j\right)}{\gamma_{n-L}\left(S_L^2 \nu\right)} \right)^2.
\end{aligned}$$

Then as $\frac{p_{n-L}\left(S_L^2 \nu, b_j\right)}{\gamma_{n-L}\left(S_L^2 \nu\right)}$ is independent of the weights, and $\alpha_{m,j} \to 0$ for $j > L$,

$$\limsup_{m \to \infty} \gamma_n(\mu_m)^{-2} \leq \gamma_{n-L}\left(S_L^2 \nu\right)^{-2}. \tag{2.17}$$

Let $\mathcal{J}$ be a compact interval containing $[-1, 1]$ and all the $\{b_j\}$. Observe that $\frac{p_n(\mu_m, t)}{\gamma_n(\mu_m)}$ is a monic polynomial of degree n with zeros in the interval $\mathcal{J}$. Since $\mathcal{J}$ is independent of m, we can by a compactness argument, choose a subsequence $\left\{\mu_{m_k}\right\}_{k=1}^{\infty}$ and a monic polynomial U of degree n such that

$$\lim_{k \to \infty} \frac{p_n\left(\mu_{m_k}, t\right)}{\gamma_n\left(\mu_{m_k}\right)} = U(t)$$

uniformly in compact subsets of $\mathbb{C}$. Next,

$$\begin{aligned}
&\gamma_n\left(\mu_{m_k}\right)^{-2} \\
= &\int\left(\frac{p_n\left(\mu_{m_k},t\right)}{\gamma_n\left(\mu_{m_k}\right)}\right)^2 d\nu\left(t\right)+\sum_{j=1}^{M}\alpha_{m_k,j}\left(\frac{p_n\left(\mu_{m_k},b_j\right)}{\gamma_n\left(\mu_{m_k}\right)}\right)^2 \\
\geq &\left(\min_{1\leq j\leq L}\alpha_{m_k,j}\right)\sum_{j=1}^{L}\left(\frac{p_n\left(\mu_{m_k},b_j\right)}{\gamma_n\left(\mu_{m_k}\right)}\right)^2
\end{aligned}$$

so that

$$\begin{aligned}
\sum_{j=1}^{L}U\left(b_j\right)^2 &= \lim_{k\to\infty}\sum_{j=1}^{L}\left(\frac{p_n\left(\mu_{m_k},b_j\right)}{\gamma_n\left(\mu_{m_k}\right)}\right)^2 \\
&\leq \limsup_{k\to\infty}\left[\min_{1\leq j\leq L}\alpha_{m_k,j}\right]^{-1}\limsup_{m\to\infty}\gamma_n\left(\mu_{m_k}\right)^{-2} \\
&\leq 0\left(\gamma_{n-L}^{-2}\left(S_L^2 d\nu\right)\right)=0,
\end{aligned}$$

by (2.17). So $U=S_L R$, for some monic polynomial R of degree $n-L$. Next, as $\mu_{m_k}\geq\nu$,

$$\begin{aligned}
&\liminf_{k\to\infty}\gamma_n\left(\mu_{m_k}\right)^{-2} \\
= &\liminf_{k\to\infty}\int_{-1}^{1}\left(\frac{p_n\left(\mu_{m_k},t\right)}{\gamma_n\left(\mu_{m_k}\right)}\right)^2 d\mu_{m_k}\left(t\right) \\
\geq &\liminf_{k\to\infty}\int_{-1}^{1}\left(\frac{p_n\left(\mu_{m_k},t\right)}{\gamma_n\left(\mu_{m_k}\right)}\right)^2 d\nu\left(t\right) \\
= &\int_{-1}^{1}R^2\left(t\right)S_L^2\left(t\right)d\nu\left(t\right) \\
\geq &\gamma_{n-L}\left(S_L^2\nu\right)^{-2}.
\end{aligned}$$

Together with (2.17), this gives

$$\lim_{k\to\infty}\gamma_n\left(\mu_{m_k}\right)^{-2}=\gamma_{n-L}\left(S_L^2\nu\right)^{-2}.$$

Since every subsequence of $\{\mu_m\}$ contains such a subsequence and the limit is independent of the subsequence, we deduce that

$$\lim_{m\to\infty}\gamma_n\left(\mu_m\right)^{-2}=\gamma_{n-L}\left(S_L^2\nu\right)^{-2}. \tag{2.18}$$

Next, let

$$\pi_m\left(t\right)=\frac{p_n\left(\mu_m,t\right)}{\gamma_n\left(\mu_m\right)}-\frac{S_L\left(t\right)p_{n-L}\left(S_L^2\nu,t\right)}{\gamma_{n-L}\left(S_L^2\nu\right)}.$$

We have

$$\begin{aligned}
&\int \left(\frac{p_n(\mu_m, t)}{\gamma_n(\mu_m)} - \frac{S_L(t)\, p_{n-L}\left(S_L^2 \nu, t\right)}{\gamma_{n-L}\left(S_L^2 \nu\right)} \right)^2 d\mu_m(t) \\
= \;& \frac{1}{\gamma_n(\mu_m)^2} - \frac{2}{\gamma_n(\mu_m)\, \gamma_{n-L}\left(S_L^2 \nu\right)} \frac{\gamma_{n-L}\left(S_L^2 \nu\right)}{\gamma_n(\mu_m)} \\
& + \int_{-1}^{1} \left(\frac{S_L(t)\, p_{n-L}\left(S_L^2 \nu, t\right)}{\gamma_{n-L}\left(S_L^2 \nu\right)} \right)^2 d\nu(t) + \sum_{j=L+1}^{M} \alpha_{m,j} \left(\frac{S_L(b_j)\, p_{n-L}\left(S_L^2 \nu, b_j\right)}{\gamma_{n-L}\left(S_L^2 \nu\right)} \right)^2 \\
= \;& -\frac{1}{\gamma_n(\mu_m)^2} + \frac{1}{\gamma_{n-L}\left(S_L^2 d\nu\right)^2} + \sum_{j=L+1}^{M} \alpha_{m,j} \left(\frac{S_L(b_j)\, p_{n-L}\left(S_L^2 \nu, b_j\right)}{\gamma_{n-L}\left(S_L^2 \nu\right)} \right)^2 \to 0,
\end{aligned}$$

as $m \to \infty$, by (2.18) and as π_m is a difference of monic polynomials of degree n, with all zeros in a compact set independent of m. Again, as the monic polynomials are of degree n and all have zeros in a fixed compact set, and as the L_2 and L_∞ norms are equivalent on polynomials of degree $\leq n$, so

$$\lim_{m \to \infty} \left(\frac{p_n(\mu_m, t)}{\gamma_n(\mu_m)} - \frac{S_L(t)\, p_{n-L}\left(S_L^2 \nu, t\right)}{\gamma_{n-L}\left(S_L^2 \nu\right)} \right) = 0$$

uniformly for t in compact sets of the plane. Using (2.18) again,

$$\lim_{k \to \infty} p_n(\mu_m, t) = S_L(t)\, p_{n-L}\left(S_L^2 \nu, t\right).$$

(b) Here we again choose a subsequence $\left\{\mu_{m_k}\right\}_{k=1}^{\infty}$ and a monic polynomial U of degree n such that uniformly for t in compact sets,

$$\lim_{k \to \infty} \frac{p_n\left(\mu_{m_k}, t\right)}{\gamma_n\left(\mu_{m_k}\right)} = U(t).$$

Next,

$$\begin{aligned}
1 &= \int p_n^2\left(\mu_{m_k}, t\right) d\mu_{m_k} \\
&\geq \sum_{j=1}^{L} \alpha_{m,j} p_n^2\left(\mu_{m_k}, b_j\right) \\
&\geq \left(\min_{1 \leq j \leq L} \alpha_{m,j} \right)^2 \gamma_n\left(\mu_{m_k}\right)^2 \sum_{j=1}^{L} \left(\frac{p_n\left(\mu_{m_k}, b_j\right)}{\gamma_n\left(\mu_{m_k}\right)} \right)^2
\end{aligned}$$

so

$$\gamma_n\left(\mu_{m_k}\right)^2 \sum_{j=1}^{L} \left(U(b_j) + o(1)\right)^2 = o(1).$$

If for some further subsequence $\gamma_n\left(\mu_{m_k}\right)$ is bounded below, then necessarily $U(b_j) = 0$ for $1 \leq j \leq L$. Since $L > n$, then $U \equiv 0$, contradicting that it is a monic polynomial of degree n. So

$$\lim_{k \to \infty} \gamma_n\left(\mu_{m_k}\right) = 0$$

and hence uniformly for t in compact subsets of $\mathbb{C}$,

$$\lim_{n \to \infty} p_n\left(\mu_{m_k}, t\right) = 0.$$

Since the limit is independent of the subsequence, it holds for the full sequence. ■

3. Varying the Weights

In this section, we study how $\Gamma = \left(\frac{p_n(\mu,y)}{p_n(\nu,y)}\right)^2$ varies as each α_j does. Throughout, we assume that

$$\mu = \nu + \sum_{j=1}^{L} \alpha_j \delta_{b_j}, \tag{3.1}$$

where all $\alpha_j \geq 0$ and all b_j are real. We use Theorem 2.2, that is, when all $\alpha_j > 0$,

$$\Gamma = \left(\frac{p_n(\mu,y)}{p_n(\nu,y)}\right)^2 = \frac{\det\left(\mathbf{D}^{-1} + \mathbf{K}_n + XY^T\right)^2}{\det\left(\mathbf{D}^{-1} + \mathbf{K}_n\right)\det\left(\mathbf{D}^{-1} + \mathbf{K}_{n+1}\right)}, \tag{3.2}$$

with the notation (2.1) to (2.5). Since α_j appears only in $\mathbf{D}$, and not in $\mathbf{K}_n, X, Y$, we see that Γ is a rational function of each α_j. Note that Γ is well defined and finite, even if some α_j are 0, even though $\mathbf{D}^{-1}$ is no longer defined.

Theorem 3.1
Let ν be a positive measure on $[-1,1]$, with infinitely many points in its support. Let $L \geq 1$ and $\{b_j\}_{j=1}^{L}$ be L distinct real points. Let $n \geq 1$, $y \in \mathbb{R}$ and $p_n(\nu,y) \neq 0$. Let $S > T \geq 0$. Let

$$\Gamma^*(S,T) = \sup\left\{\left(\frac{p_n(\mu,y)}{p_n(\nu,y)}\right)^2 : \mu \text{ is of form (3.1) with all } \alpha_j \in [T,S]\right\}.$$

Then we can find a measure

$$\mu^* = \nu + \sum_{j=1}^{L} \alpha_j^* \delta_{b_j}$$

with all $\alpha_j^* \in \{S,T\}$ attaining the sup $\Gamma^*(S,T)$. Moreover, if Γ given by (3.2) is not a constant function of any α_j, the sup is attained only for such measures.

Remarks
(a) For the case $L = 2$, $\{b_1, b_2\} = \{-1, 1\}$, $y = 0$, n even, and ν symmetric, this theorem appears essentially in [23]. There a study was undertaken when ν is in the Nevai class, and in more detail for the ultraspherical weight $\nu'(t) = \left(1-t^2\right)^{\alpha}, \alpha > -1$. For $\alpha = -\frac{1}{2}$, there was exactly one extremal measure when considering weights (α_1, α_2) in the simplex $\alpha_1 + \alpha_2 \leq S$, with $\alpha_1 = \alpha_2 = S/2$. For $\alpha > -\frac{1}{2}$, there could be one or two extremal measures depending on the choice of S. Moreover, for the analogue of $\Gamma^*(\infty, 0)$ above, it was found that as $n \to \infty$,

$$\Gamma^*(\infty,0) = 1 + \frac{2(\alpha+1)}{(n+\alpha)^2} + O\left(n^{-3}\right),$$

so is larger than 1 for large enough n, but decays to 1 as $n \to \infty$.
(b At first it seems reasonable that either for all j, $\alpha_{m,j} \to 0$ or for all j, $\alpha_{m,j} \to \infty$. However, an analysis in the case $L = 2, 1 < b_1 < b_2$ suggested that it is possible that $\alpha_{m,1} \to 0$ and $\alpha_{m,2} \to \infty$.

To prove this theorem, we investigate Γ as a function of a single α_j. Given a matrix B, we let $B(j;k)$ denote the matrix obtained from B by removing row j and column k. In addition, if B is a matrix such that its (j,j) entry is $\alpha_j^{-1}+d$, then we let $B^{\#}(j)$ denote the same matrix B but with the (j,j) entry replaced by d. Thus we are just subtracting α_j^{-1} from the (j,j) entry. Throughout, we assume that n,y and ν are as in Theorem 3.1. We begin with some elementary manipulation. In Lemmas 3.2 to 3.5 we assume that $p_n(\nu,y)\neq 0$.

Lemma 3.2
Let μ be given by (3.1) and Γ be given by (3.2). Fix $1\leq j\leq L$. Fix $\alpha_k>0$ for $k\neq j$.
(a) Γ is a rational function of type $(2,2)$ in α_j. More precisely,

$$\Gamma=\frac{\left(H_n+\alpha_jJ_n\right)^2}{\left(F_n+\alpha_jG_n\right)\left(F_{n+1}+\alpha_jG_{n+1}\right)}, \tag{3.3}$$

where

$$\begin{aligned} F_n &= \det\mathbf{C}_n(j;j)>0; G_n=\det\mathbf{C}_n^{\#}(j)>0;\\ H_n &= \det\left(\mathbf{C}_n+Y^TX\right)(j;j); J_n=\det\left(\mathbf{C}_n+Y^TX\right)^{\#}(j). \end{aligned} \tag{3.4}$$

Here F_n,G_n,H_n,J_n do not depend on α_j, but may depend on α_k for $k\neq j$.
(b)

$$F_nG_{n+1}-G_nF_{n+1}\geq 0. \tag{3.5}$$

Moreover,

$$\frac{F_n+\alpha_jG_n}{F_{n+1}+\alpha_jG_{n+1}}$$

is a monotone decreasing function of $\alpha_j\in[0,\infty)$.
(c) If $F_nG_{n+1}-G_nF_{n+1}>0$, then

$$\frac{\partial\Gamma}{\partial\alpha_j} = \frac{1}{F_nG_{n+1}-F_{n+1}G_n}\left\{-\frac{\left(H_nG_{n+1}-F_{n+1}J_n\right)^2}{\left(F_{n+1}+\alpha_jG_{n+1}\right)^2}+\frac{\left(H_nG_n-F_nJ_n\right)^2}{\left(F_n+\alpha_jG_n\right)^2}\right\}. \tag{3.6}$$

In addition

$$\frac{\partial\Gamma}{\partial\alpha_j}_{|\alpha_j=0}=-\frac{H_n^2}{F_n^2F_{n+1}^2}\left(G_nF_{n+1}+F_nG_{n+1}\right)+\frac{2J_nH_n}{F_nF_{n+1}}, \tag{3.7}$$

and as $\alpha_j\to\infty$,

$$\frac{\partial\Gamma}{\partial\alpha_j} = \frac{1}{\alpha_j^2}\left\{\frac{J_n^2}{G_n^2G_{n+1}^2}\left(F_nG_{n+1}+G_nF_{n+1}\right)-\frac{2J_nH_n}{G_nG_{n+1}}+O\left(\frac{1}{\alpha_j}\right)\right\}. \tag{3.8}$$

Proof
(a) Recall from Theorem 2.2 that

$$\Gamma=\left(\frac{p_n(\mu,y)}{p_n(\nu,y)}\right)^2=\frac{\left(\det\left[\mathbf{C}_n+YX^T\right]\right)^2}{\left(\det\mathbf{C}_{n+1}\right)\left(\det\mathbf{C}_n\right)}.$$

Here α_j appears only in the (j,j) position in each matrix. Expanding by the jth row, we see that

$$\begin{aligned} & \det(\mathbf{C}_n) = \det(\mathbf{D}^{-1}+\mathbf{K}_n) \\ = & \ \alpha_j^{-1}\det(\mathbf{D}^{-1}+\mathbf{K}_n)(j;j) + \det(\mathbf{D}^{-1}+\mathbf{K}_n)^{\#}(j) \\ = & \ \alpha_j^{-1}F_n + G_n. \end{aligned}$$

Here $(\mathbf{D}^{-1}+\mathbf{K}_n)(j;j)$ and $(\mathbf{D}^{-1}+\mathbf{K}_n)^{\#}(j)$ are still positive definite, so $F_n, G_n > 0$. While the former is obvious, the latter follows from the fact that given

$$Z = [x_1 \ x_2 ... x_L]^T$$

with all real entries,

$$\begin{aligned} & Z^T(\mathbf{D}^{-1}+\mathbf{K}_n)^{\#}(j)\,Z \\ \geq & \ Z^T\left(\mathbf{D}^{-1}\right)^{\#}(j)\,Z = \sum_{k=1,k\neq j}^{L} \alpha_j^{-1}x_k^2 > 0 \end{aligned}$$

provided at least one of $x_1...x_{j-1}, x_{j+1}...x_L$ is non-zero. If they are all zero, but $x_j \neq 0$, we then have

$$Z^T(\mathbf{D}^{-1}+\mathbf{K}_n)^{\#}(j)\,Z = K_n(\nu, b_j, b_j)\,x_j^2 > 0.$$

So indeed, $(\mathbf{D}^{-1}+\mathbf{K}_n)^{\#}(j)$ is positive definite. Similarly,

$$\begin{aligned} & \det\left[\mathbf{D}^{-1}+\mathbf{K}_n + YX^T\right] \\ = & \ \alpha_j^{-1}\det\left(\mathbf{D}^{-1}+\mathbf{K}_n + YX^T\right)(j;j) + \det\left(\mathbf{D}^{-1}+\mathbf{K}_n + YX^T\right)^{\#}(j) \\ = & \ \alpha_j^{-1}H_n + J_n. \end{aligned}$$

Then

$$\begin{aligned} \Gamma &= \frac{\left(\alpha_j^{-1}H_n + J_n\right)^2}{\left(\alpha_j^{-1}F_{n+1} + G_{n+1}\right)\left(\alpha_j^{-1}F_n + G_n\right)} \\ &= \frac{\left(H_n + \alpha_j J_n\right)^2}{\left(F_{n+1} + \alpha_j G_{n+1}\right)\left(F_n + \alpha_j G_n\right)}. \end{aligned}$$

(b) From Theorem 2.2(c),

$$\frac{\det \mathbf{C}_n}{\det \mathbf{C}_{n+1}} = \frac{F_n + \alpha_j G_n}{F_{n+1} + \alpha_j G_{n+1}}$$

is a decreasing function of $\alpha_j \in (0,\infty)$, so for such α_j,

$$\begin{aligned} & \frac{\partial}{\partial \alpha_j}\left(\frac{F_n + \alpha_j G_n}{F_{n+1} + \alpha_j G_{n+1}}\right) \leq 0 \\ \Rightarrow & \ \frac{F_{n+1}G_n - G_{n+1}F_n}{\left(F_{n+1} + \alpha_j G_{n+1}\right)^2} \leq 0. \end{aligned}$$

(c) If $F_nG_{n+1} - G_nF_{n+1} > 0$, then the partial fraction decomposition of Γ as a rational function of α_j is

$$\begin{aligned} & \Gamma \\ = & \ \frac{J_n^2}{G_{n+1}G_n} + \frac{1}{F_nG_{n+1} - F_{n+1}G_n}\left\{\frac{\left(H_nG_{n+1} - F_{n+1}J_n\right)^2}{G_{n+1}\left(F_{n+1} + \alpha_j G_{n+1}\right)} - \frac{\left(H_nG_n - F_nJ_n\right)^2}{G_n\left(F_n + \alpha_j G_n\right)}\right\}. \end{aligned}$$

This follows by a straightforward calculation. Then

$$\frac{\partial\Gamma}{\partial\alpha_j} = \frac{1}{F_nG_{n+1}-F_{n+1}G_n}\left\{-\frac{\left(H_nG_{n+1}-F_{n+1}J_n\right)^2}{\left(F_{n+1}+\alpha_jG_{n+1}\right)^2}+\frac{\left(H_nG_n-F_nJ_n\right)^2}{\left(F_n+\alpha_jG_n\right)^2}\right\}.$$

Next

$$\begin{aligned}
\frac{\partial\Gamma}{\partial\alpha_j}_{|\alpha_j=0}
&= \frac{1}{F_nG_{n+1}-F_{n+1}G_n}\left\{-\left(H_n\frac{G_{n+1}}{F_{n+1}}-J_n\right)^2+\left(H_n\frac{G_n}{F_n}-J_n\right)^2\right\}\\
&= \frac{1}{F_nG_{n+1}-F_{n+1}G_n}\left\{H_n^2\left(\left(\frac{G_n}{F_n}\right)^2-\left(\frac{G_{n+1}}{F_{n+1}}\right)^2\right)+2J_nH_n\left(\frac{G_{n+1}}{F_{n+1}}-\frac{G_n}{F_n}\right)\right\}\\
&= \frac{1}{F_nG_{n+1}-F_{n+1}G_n}\left\{\begin{array}{c}\frac{H_n^2}{F_n^2F_{n+1}^2}\left(G_nF_{n+1}-F_nG_{n+1}\right)\left(G_nF_{n+1}+F_nG_{n+1}\right)\\ -\frac{2J_nH_n}{F_nF_{n+1}}\left(G_nF_{n+1}-F_nG_{n+1}\right)\end{array}\right\}\\
&= -\frac{H_n^2}{F_n^2F_{n+1}^2}\left(G_nF_{n+1}+F_nG_{n+1}\right)+\frac{2J_nH_n}{F_nF_{n+1}}.
\end{aligned}$$

Finally, as $\alpha_j\to\infty$,

$$\begin{aligned}
\frac{\partial\Gamma}{\partial\alpha_j}
&= \frac{1}{F_nG_{n+1}-F_{n+1}G_n}\frac{1}{\alpha_j^2}\left\{-\left(H_n-\frac{F_{n+1}}{G_{n+1}}J_n\right)^2+\left(H_n-\frac{F_n}{G_n}J_n\right)^2+O\left(\frac{1}{\alpha_j}\right)\right\}\\
&= \frac{1}{F_nG_{n+1}-F_{n+1}G_n}\frac{1}{\alpha_j^2}\left\{\begin{array}{c}J_n^2\left(\left(\frac{F_n}{G_n}\right)^2-\left(\frac{F_{n+1}}{G_{n+1}}\right)^2\right)\\ +2J_nH_n\left(\frac{F_{n+1}}{G_{n+1}}-\frac{F_n}{G_n}\right)+O\left(\frac{1}{\alpha_j}\right)\end{array}\right\}\\
&= \frac{1}{\alpha_j^2}\left\{\frac{J_n^2}{G_n^2G_{n+1}^2}\left(F_nG_{n+1}+G_nF_{n+1}\right)-\frac{2J_nH_n}{G_nG_{n+1}}+O\left(\frac{1}{\alpha_j}\right)\right\}.
\end{aligned}$$

■

We consider separately the cases where $F_nG_{n+1}-G_nF_{n+1}$ is positive or 0.

Lemma 3.3
Let $1\le j\le L$, and fix $\alpha_k>0$ for $k\ne j$. Assume

$$F_nG_{n+1}-G_nF_{n+1}>0. \tag{3.9}$$

Let $S>T>0$ and

$$\Gamma_j^*=\sup\left\{\Gamma:T\le\alpha_j\le S\right\} \tag{3.10}$$

and assume

$$\Gamma_j^*>0. \tag{3.11}$$

Then Γ_j^* is attained only when $\alpha_j = T$ or $\alpha_j = S$.

Proof

We do this in a number of steps.

Step 1: There is at most one value of $\alpha_j \in [0,\infty)$ at which $\frac{\partial \Gamma}{\partial \alpha_j} = 0$.

For $\frac{\partial \Gamma}{\partial \alpha_j} = 0$, (3.6) shows that we must have

$$\frac{(H_n G_{n+1} - F_{n+1} J_n)^2}{(F_{n+1} + \alpha_j G_{n+1})^2} = \frac{(H_n G_n - F_n J_n)^2}{(F_n + \alpha_j G_n)^2}.$$

If either $H_n G_{n+1} - F_{n+1} J_n$ or $H_n G_n - F_n J_n$ is non-zero, then both must be non-zero (recall $F_n, G_n > 0$), and

$$\frac{(F_n + \alpha_j G_n)^2}{(F_{n+1} + \alpha_j G_{n+1})^2} = \frac{(H_n G_n - F_n J_n)^2}{(H_n G_{n+1} - F_{n+1} J_n)^2}. \tag{3.12}$$

Taking square roots,

$$\frac{F_n + \alpha_j G_n}{F_{n+1} + \alpha_j G_{n+1}} = \pm \frac{H_n G_n - F_n J_n}{H_n G_{n+1} - F_{n+1} J_n}. \tag{3.13}$$

The left-hand side is positive and is a decreasing rational function of $\alpha_j \in [0,\infty)$, by Lemma 3.2(b). It is also non-constant because of (3.9). So only one choice of the right-hand side works. Thus there is at most one α_j in $[0,\infty)$ where Γ has a critical point.

Next suppose $H_n G_{n+1} - F_{n+1} J_n = H_n G_n - F_n J_n = 0$. If first one of J_n, H_n is non-zero, then both must be non-zero because F_n, G_n are positive. In this case,

$$\frac{H_n}{J_n} = \frac{F_n}{G_n} = \frac{F_{n+1}}{G_{n+1}},$$

contradicting our hypothesis (3.9. The last remaining possibility is that $J_n = H_n = 0$. But then $\Gamma = 0$ for all $\alpha_j \in (0,\infty)$, contradicting our hypothesis (3.11).

Step 2: Monotonicity properties

Because there is at most one critical point in $[0,\infty)$, there are the following possibilities:

(i) Γ is strictly decreasing in $(0,\infty)$.

(ii) Γ is strictly increasing in $(0,\infty)$.

(iii) Γ decreases strictly to a local minimum and then increases thereafter.

(iv) Γ increases to a local maximum and then decreases thereafter.

In the first three cases, the conclusion of the lemma follows. We proceed to show that (iv) is not possible:

Step 3: If $\frac{\partial \Gamma}{\partial \alpha_j}_{|\alpha_j = 0} > 0$, then Γ is increasing in $(0,\infty)$.

First (3.7) shows that $H_n \neq 0$. Then

$$\frac{\partial \Gamma}{\partial \alpha_j}_{|\alpha_j=0} = -\frac{2H_n^2}{F_n F_{n+1}} \left(\frac{1}{2} \left[\frac{G_n}{F_n} + \frac{G_{n+1}}{F_{n+1}} \right] - \frac{J_n}{H_n} \right).$$

Since $F_n, F_{n+1} > 0$ and $H_n \neq 0$, our hypothesis forces

$$\frac{1}{2} \left[\frac{G_n}{F_n} + \frac{G_{n+1}}{F_{n+1}} \right] < \frac{J_n}{H_n}. \tag{3.14}$$

We next use the inequality between geometric, arithmetic and harmonic means. Recall that if α, β are positive real numbers,

$$\frac{1}{2}(\alpha+\beta) \geq \sqrt{\alpha\beta} \geq \frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}},$$

and we have strict inequality unless $\alpha = \beta$. Since we are assuming $\frac{G_n}{F_n} < \frac{G_{n+1}}{F_{n+1}}$, (3.14) gives the strict inequality

$$\frac{J_n}{H_n} > \frac{2}{\frac{F_n}{G_n}+\frac{F_{n+1}}{G_{n+1}}}.$$

Then we cannot have $J_n = 0$, so (recall $F_n, F_{n+1}, G_n, G_{n+1} > 0$),

$$\frac{F_n}{G_n}+\frac{F_{n+1}}{G_{n+1}} > 2\frac{H_n}{J_n}. \tag{3.15}$$

Then from (3.8),

$$\begin{aligned} &\frac{\partial \Gamma}{\partial \alpha_j} \\ &= \frac{1}{\alpha_j^2}\frac{J_n^2}{G_nG_{n+1}}\left\{\frac{F_n}{G_n}+\frac{F_{n+1}}{G_{n+1}}-\frac{2H_n}{J_n}+O\left(\frac{1}{\alpha_j}\right)\right\} > 0 \end{aligned} \tag{3.16}$$

for large enough α_j. So Γ is an increasing function of α_j for large enough α_j. Since $\frac{\partial \Gamma}{\partial \alpha_j}_{|\alpha_j=0} > 0$, we claim that $\frac{\partial \Gamma}{\partial \alpha_j} > 0$ in $(0,\infty)$. Otherwise it would have to have both a local maximum and minimum, contradicting that it has at most one critical point. So Γ is increasing in $(0,\infty)$. Thus the case (iv) above is not possible and the lemma is proved. ■

Lemma 3.4
Let $1 \leq j \leq L$, and fix $\alpha_k > 0$ for $k \neq j$. Assume

$$F_nG_{n+1} - G_nF_{n+1} = 0. \tag{3.17}$$

Then Γ is constant for $\alpha_j \in [0,\infty)$, and moreover, for all $\alpha_j > 0$, $p_n(\mu, x)$ is the same polynomial, and $p_n(\mu, b_j) = 0$.
Proof
First recall from (2.14) and (3.4) that

$$\left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2 = \frac{\det \mathbf{C}_n}{\det \mathbf{C}_{n+1}} = \frac{F_n+\alpha_jG_n}{F_{n+1}+\alpha_jG_{n+1}}.$$

Our hypothesis shows that for all α_j,

$$\left(\frac{\gamma_n(\mu)}{\gamma_n(\nu)}\right)^2 = \frac{G_n}{G_{n+1}}.$$

So $\gamma_n(\mu)$ is constant in α_j. Now let us consider two such measures. Let $0 \leq \Delta < \beta$ and let μ_Δ be the measure μ where $\alpha_j = \Delta$ and μ_β be the measure μ where $\alpha_j = \beta$. We already know that

$$\gamma_n(\mu_\Delta) = \gamma_n(\mu_\beta).$$

Then

$$\begin{aligned} & \int \left(p_n\left(\mu_\Delta, x\right) - p_n\left(\mu_\beta, x\right)\right)^2 d\mu_\Delta(x) \\ = \quad & 1 - 2\frac{\gamma_n\left(\mu_\beta\right)}{\gamma_n\left(\mu_\Delta\right)} + \int p_n\left(\mu_\beta, x\right)^2 d\mu_\Delta(x) \\ = \quad & 1 - 2 + 1 + p_n\left(\mu_\beta, b_j\right)^2 (\Delta - \beta) \\ = \quad & p_n\left(\mu_\beta, b_j\right)^2 (\Delta - \beta). \end{aligned}$$

The left-hand side is nonnegative, so the right-hand side must be also. However, $\Delta - \beta < 0$. Then necessarily $p_n\left(\mu_\beta, b_j\right) = 0$, and moreover,

$$\int \left(p_n\left(\mu_\Delta, x\right) - p_n\left(\mu_\beta, x\right)\right)^2 d\mu_\Delta(x) = 0,$$

so that $p_n\left(\mu_\Delta, x\right) \equiv p_n\left(\mu_\beta, x\right)$. From the definition (3.2) of Γ, it is constant for $\alpha_j \in [0, \infty)$. ■

Proof of Theorem 3.1
Suppose first $T > 0$. Choose

$$\mu_m = \nu + \sum_{j=1}^{L} \alpha_{m,j} \delta_{b_j}$$

with

$$\lim_{m\to\infty} \left(\frac{p_n\left(\mu_m, y\right)}{p_n(\nu, y)}\right)^2 = \Gamma^*(S, T).$$

By passing to a subsequence, we may assume that for $1 \leq j \leq L$,

$$\lim_{m\to\infty} \alpha_{m,j} = \alpha_j^*.$$

Let

$$\mu^* = \nu + \sum_{j=1}^{L} \alpha_j^* \delta_{b_j}.$$

Continuity of orthonormal polynomials in the measure (for fixed degree) ensures that

$$\left(\frac{p_n\left(\mu^*, y\right)}{p_n(\nu, y)}\right)^2 = \Gamma^*(S, T).$$

From the previous two lemmas, if Γ is not constant in α_j, then necessarily $\alpha_j^* = S$ or T. If Γ is constant in α_j, we then can choose $\alpha_j^* = S$ or T. Finally if $T = 0$, we can apply the above with a sequence of values of T decreasing to 0. ■

4. Proof of Theorems 1.1 to 1.3

We begin with

The Proof of Theorem 1.3 (a)
For the case where $p_n(\nu, y) \neq 0$, this already follows from Theorem 3.1. Now

suppose that $p_n(\nu, y) = 0$. Choose a sequence $\{y_k\}$ converging to y, but with $p_n(\nu, y_k) \neq 0$ for $k \geq 1$. By Theorem 3.1, for each k, there is a measure

$$\mu_k = \nu + \sum_{j=1}^{L} \alpha_{j,k} \delta_{b_j}$$

with all $\alpha_{j,k} \in \{S, T\}$ and

$$\left(\frac{p_n(\mu_k, y_k)}{p_n(\nu, y_k)} \right)^2 = \sup \left\{ \left(\frac{p_n(\mu, y_k)}{p_n(\nu, y_k)} \right)^2 : \mu \text{ of form (3.1) with all } \alpha_j \in [S, T] \right\}.$$

Since there are at most 2^L choices for $(\alpha_{j,k}, ...\alpha_{L,k})$, by passing to a subsequence we may assume that

$$\alpha_{j,k} = \alpha_{j,1} \text{ for all } j, k.$$

Then $\mu_k = \mu_1$ for all k, so that cancelling $p_n(\nu, y_k)$,

$$p_n(\mu_1, y_k)^2 = \sup \left\{ p_n(\mu, y_k)^2 : \mu \text{ of form (3.1) with all } \alpha_j \in [S, T] \right\}.$$

But then if μ is any measure of the form (3.1) with all $\alpha_j \in [S, T]$,

$$\begin{aligned} p_n(\mu, y)^2 &= \lim_{k \to \infty} p_n(\mu, y_k)^2 \\ &\leq \lim_{k \to \infty} p_n(\mu_1, y_k)^2 = p_n(\mu_1, y)^2. \end{aligned}$$

It follows that μ_1 is a measure of the form (3.1) with all weights S or T, and

$$p_n(\mu_1, y)^2 = \sup \left\{ p_n(\mu, y)^2 : \mu \text{ of form (3.1) with all } \alpha_j \in [S, T] \right\}.$$

■

The Proof of Theorem 1.3(b)

First assume that $p_n(\nu, y) \neq 0$. Choose measures

$$\mu_m = \nu + \sum_{j=1}^{L} \alpha_{m,j} \delta_{b_j}, m \geq 1,$$

with

$$\lim_{m \to \infty} p_n(\mu_m, y)^2 = \Omega^* = \sup \left\{ p_n(\mu, y)^2 : \mu \text{ is of form (3.1) with all } \alpha_j \geq 0 \right\}.$$

For $1 \leq j \leq L$, we may choose a subsequence of $\{\alpha_{m,j}\}_{m=1}^{\infty}$ having limit $\alpha_j^* \in [0, \infty]$. By changing our notation, we may assume that for each j,

$$\lim_{m \to \infty} \alpha_{m,j} = \alpha_j^*.$$

We can separate into three sets of indices: $j \in \mathcal{J}_1$ if $\alpha_j^* = 0$; $j \in \mathcal{J}_2$ if $\alpha_j^* = \infty$; $j \in \mathcal{J}_3$ if $0 < \alpha_j^* < \infty$. If $j \in \mathcal{J}_3$, then by Theorem 3.1, Γ above is a constant function of $\alpha_j \in [0, \infty)$, so remains the same as $\alpha_j \to \infty$. So we may insert all the indices in $\mathcal{J}_3$ into those in $\mathcal{J}_2$, and dispense with $\mathcal{J}_3$. Thus all the α_j^* may be assumed to be 0 or ∞. Next if the expanded $\mathcal{J}_2$ contains more than n indices, Lemma 2.3(b) shows that

$$\Omega^* = \lim_{m \to \infty} p_n(\mu_m, y) = 0,$$

contradicting that $\Omega^* \geq p_n(\nu, y)^2$. So $\mathcal{J}_2$ has at most n indices. Write

$$\mu_m = \nu + \sum_{j \in \mathcal{J}_1} \alpha_{m,j} \delta_{b_j} + \sum_{j \in \mathcal{J}_2} \alpha_{m,j} \delta_{b_j}. \tag{4.1}$$

If S is the monic polynomial of degree $M (\leq n)$ say, whose zeros are those b_j with $j \in \mathcal{J}_2$, Lemma 2.3(a) shows that

$$\lim_{m \to \infty} p_n(\mu_m, t)^2 = S_M^2(t) \, p_{n-M}\left(S_M^2 \nu, t\right)$$

uniformly for t in compact subsets of the plane. In particular,

$$\Omega^* = \lim_{m \to \infty} p_n(\mu_m, y)^2 = S_M^2(y) \, p_{n-M}^2\left(S_M^2 \nu, y\right)$$

so that

$$\Omega^* \leq \sup S_J^2(y) p_{n-J}^2\left(S_J^2 \nu, y\right)$$

where the sup is taken over all $0 \leq J \leq n$ and monic polynomials S_J of degree $\leq \min\{n, L\}$ with zeros in $\{b_j\}_{j=1}^L$.

For the converse direction, let S_J be a monic polynomial of degree $J \leq n$, with zeros $b_j, j \in \mathcal{J}_2$ say. We let

$$\mu_m = \nu + m \sum_{j \in \mathcal{J}_2} \delta_{b_j}, m \geq 1, \tag{4.2}$$

and apply Lemma 2.3 (a) to deduce that

$$\Omega^* \geq \lim_{m \to \infty} p_n(\mu_m, y)^2 = S_J^2(y) \, p_{n-J}^2\left(S_J^2 \nu, y\right).$$

As S_J is any such polynomial,

$$\Omega^* = \sup S_J^2(y) p_{n-J}^2\left(S_J^2 \nu, y\right)$$

where the sup is taken over all $0 \leq J \leq n$ and monic polynomials S_J of degree $\leq \min\{n, L\}$ with zeros in $\{b_j\}_{j=1}^L$.

We now show that the conclusion of (b) remains valid if $p_n(\nu, y) = 0$. Choose a sequence $\{y_k\}$ converging to y, but with $p_n(\nu, y_k) \neq 0$ for $k \geq 1$. For each k, there is a polynomial S_{J_k} as above such that

$$\begin{aligned} & \sup\left\{p_n^2(\mu, y_k) : \mu \text{ is of form (3.1) with all } \alpha_j \geq 0\right\} \\ = \; & S_{J_k}^2(y_k) \, p_{n-J_k}^2\left(S_{J_k}^2 \nu, y\right). \end{aligned}$$

There are only finitely many monic polynomials of degree $J \leq n$ with all distinct zeros in the finite set $\{b_j\}_{j=1}^L$. So we may choose a further subsequence (which we denote in the same way) such that $J_k = J$ and $S_{J_k} = S_J$ for all k. Then for any μ of the form (3.1),

$$\begin{aligned} p_n^2(\mu, y) &= \lim_{k \to \infty} p_n^2(\mu, y_k) \\ &\leq \limsup_{k \to \infty} S_J^2(y_k) \, p_{n-J}^2\left(S_J^2 \nu, y_k\right) \\ &= S_J^2(y) \, p_{n-J}^2(S_J^2 \nu, y). \end{aligned}$$

So

$$\Omega^* = \sup\left\{p_n^2(\mu, y) : \mu \text{ is of form (3.1) with all } \alpha_j \geq 0\right\} \leq S_J^2(y) \, p_{n-J}^2\left(S_J^2 \nu, y\right).$$

The opposite inequality follows from Lemma 2.3, by considering $\{\mu_m\}$ of the form (4.2). ■

We turn to

The Proof of Theorem 1.2(a)

Let μ be the given measure. Let $\omega = \mu - \nu$. The idea here is to approximate ω by discrete measures with masspoints in the given set $\mathcal{B}_L$. Assume that the points in $\mathcal{B}_L \cap [1, \infty)$ are

$$1 \leq b_1^+ < b_2^+ < ...$$

and in $\mathcal{B}_L \cap (-\infty, -1]$ are

$$-1 \geq b_1^- > b_2^- > ...$$

The $\left\{b_j^+\right\}, \left\{b_j^-\right\}$ depend on L, but we do not explicitly display this dependence. Choose discrete measures

$$\rho_L^+ = \sum_j \delta_{b_j^+} \int_{[b_j^+, b_{j+1}^+)} d\mu$$

and

$$\rho_L^- = \sum_j \delta_{b_j^-} \int_{(b_{j+1}^-, b_j^-]} d\mu.$$

Moreover, if there are mass points at some $b_j^{\pm}$, we include them in the integral. Because of our hypotheses on the spacing of the $\left\{b_j^{\pm}\right\}$, it follows that $\rho_L^+ + \rho_L^-$ converges weakly to ω as $L \to \infty$. Hence as $L \to \infty$, all the moments of $\nu + \rho_L^+ + \rho_L^-$ converge to those of μ, so also

$$\lim_{L\to\infty} p_n\left(\nu + \rho_L^+ + \rho_L^-, y\right) = p_n\left(\mu, y\right).$$

By Theorem 1.3(b),

$$p_n^2\left(\nu + \rho_L^+ + \rho_L^-, y\right) \leq M_L, L \geq 1,$$

so

$$p_n^2\left(\mu, y\right) \leq \limsup_{L\to\infty} M_L.$$

■

The Proof of Theorem 1.2(b)

Let $\omega_1 = (\mu - \nu)_{|[1,\infty)}$ and $\omega_2 = (\mu - \nu)_{|(-\infty,-1]|}$. If ω_1 has at least $n+1$ points in its support, we may form its Gauss quadrature of order $n+1$. Write this in the form

$$\int P\left(t\right) d\omega_1^G\left(t\right) = \sum_{j=1}^{n+1} \lambda_j^+ P\left(b_j^+\right)$$

for polynomials P of degree $\leq 2n+1$. Here all $b_j^+ > 1$. Similarly, If ω_2 has at least $n+1$ points in its support, we may form its Gauss quadrature of order $n+1$:

$$\int P\left(t\right) d\omega_2^G\left(t\right) = \sum_{j=1}^{n+1} \lambda_j^- P\left(b_j^-\right)$$

for polynomials P of degree $\leq 2n+1$. Here all $b_j^- < -1$. In this case, we let $\mathcal{B}_G$ be the set of all the $\left\{b_j^-\right\}, \left\{b_j^+\right\}$. If one or both of ω_1, ω_2 have less than $n+1$ points in their support, we just take the mass points in ω_1 and/or ω_2. Then for polynomials P of degree $\leq 2n+1$,

$$\int P(t)\, d\left(\nu+\omega_1^G+\omega_2^G\right)(t) = \int P(t)\, d\mu(t).$$

It follows that μ and $\nu+\omega_1^G+\omega_2^G$ have the same orthogonal polynomial of degree n, as it is determined by the first $2n+1$ moments of the measure. Here $\nu+\omega_1^G+\omega_2^G$ has the form to which we can apply Theorem 1.3(b), where the $\left\{b_j^{\pm}\right\}$ are the points in $\mathcal{B}_G$. It follows from Theorem 1.3(b) that

$$p_n(\mu,y)^2 \leq \sup S_J^2(y)\, p_{n-J}\left(S_J^2\nu, y\right)$$

where the sup is taken over all $0 \leq J \leq n$ and all monic polynomials S_J of degree J, with distinct zeros in $\mathcal{B}_G$. ■

The Proof of Theorem 1.1

Assume that $\mu_{|(-1,1)} = \nu$ and supp$[\mu] \subseteq \mathcal{K}$. Then we can find a sequence of measures $\{\mu_L\}_{L\geq 1}$, where

$$\mu_L = \nu + \sum_{j=1}^{L} \alpha_{L,j}\delta_{b_{L,j}},$$

and all $\{b_{L,j}\}_{j,L}$ lie in $\mathcal{K}\backslash(-1,1)$, which contains the support of $\mu-\nu$, while

$$\lim_{L\to\infty} \int t^j d\mu_L(t) = \int t^j d\mu(t), 0 \leq j \leq 2n+1.$$

(Here we want the masspoints to lie in $\mathcal{K}\backslash(-1,1)$, which is more restrictive than lying in $\mathbb{R}\backslash(-1,1)$). For compactly supported $\mu-\nu$, this follows from the classic fact that pure jump measures are weakly dense in the set of measures with given compact support. When $\mu-\nu$ has non-compact support, we can take its intersection with a growing sequence of compact intervals. By Theorem 1.3, for each $L \geq 1$,

$$p_n^2(\mu_L, y) \leq \sup S_J^2(y)\, p_{n-J}^2\left(S_J^2\nu, y\right),$$

where the supremum is taken over all $0 \leq J \leq n$ and monic polynomials S_J of degree J with distinct zeros in $\mathcal{K}\backslash(-1,1)$. Denote the latter supremum by Λ. It is finite as it is bounded above by $K_{n+1}(\nu,y,y)$. Then also

$$p_n^2(\mu,y) = \lim_{L\to\infty} p_n^2(\mu_L,y) \leq \Lambda.$$

Thus as μ is any such measure,

$$\sup\left\{p_n^2(\mu,y) : \mu_{|(-1,1)} = \nu \text{ and supp}[\mu] \subseteq \mathcal{K}\right\} \leq \Lambda. \tag{4.3}$$

For the opposite inequality, let $\varepsilon > 0, J \leq n$ and choose a monic polynomial S_J with distinct zeros in $\mathcal{K}\backslash(-1,1)$ such that

$$S_J^2(y)\, p_{n-J}^2\left(S_J^2\nu, y\right) \geq \Lambda - \varepsilon.$$

If $\{b_j\}_{j=1}^L$ are the zeros of S_J, choose

$$\mu_m = \nu + m\sum_{j=1}^{L} \delta_{b_j}, m \geq 1.$$

Then Lemma 2.3 shows that

$$\lim_{m\to\infty} p_n\left(\mu_m, y\right)^2 = S_J^2\left(y\right) p_{n-J}^2\left(S_J^2\nu, y\right) \geq \Lambda - \varepsilon.$$

So we have also the converse inequality to (4.3). ■

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References

[1] M. Alexis, A. Aptekarev, S.Denisov, *Continuity of weighted operators, Muckenhoupt A_p weights, and Steklov problem for orthogonal polynomials,* Int. Math. Res. Not. IMRN 2022, no. 8, 5935–5972.

[2] R. Alvarez-Nodarse, F. Marcellán, J. Petronilho, *WKB approximation and Krall-type orthogonal polynomials,* Acta Applicandae Mathematicae, 54 (1998), no. 1, 27–58.

[3] M.U. Ambroladze, *On the possible growth of polynomials orthogonal with a continuous positive weight,* Mat. Zametki, 45(1989), 99-101.

[4] M.U. Ambroladze, *On the possible rate of growth of polynomials that are orthogonal with a continuous positive weight.* Math. USSR-Sb. 72 (1992), 311–331.

[5] A. Aptekarev, S. Denisov, D. Tulyakov, *Steklov's problem of estimating the growth of orthogonal polynomials*, Proc. Steklov Inst. Math. 289 (2015), 72–95.

[6] A. Aptekarev, S. Denisov, D. Tulyakov, *On a problem by Steklov*, J. Amer. Math. Soc. 29 (2016), 1117–1165.

[7] J. Arvesu, F. Marcellan, R. Alvarez-Nodarse, *On a Modification of the Jacobi Linear Functional: Asymptotic Properties and Zeros of the Corresponding Orthogonal Polynomials*, Acta Applicandae Mathematicae, 71(2002), 127-158.

[8] V.M. Badkov, *The asymptotic behavior of orthogonal polynomials.* (Russian) Mat. Sb. (N.S.) 109(151) (1979), no. 1, 46–59.

[9] S.S. Bonan and D. Clark, *Estimates of the Orthogonal Polynomials with Weight* $\exp\left(-x^m\right)$, *m an even positive integer*, J. Approx. Theory, 46(1986), 408-410.

[10] A.M. Delgado, L. Fernandez, T.E. Perez, M.A. Pinar, Y. Xu, *Orthogonal Polynomials in Several Variables for Measures with Mass Points*, Numer. Algorithms, 55(2010), 245-264.

[11] S. Denisov, *Remark on the formula by Rakhmanov and Steklov's conjecture,* J. Approx. Theory 205 (2016), 102–113.

[12] S. Denisov, *On the size of the polynomials orthonormal on the unit circle with respect to a measure which is a sum of the Lebesgue measure and P point masses*, Proc. Amer. Math. Soc. 144 (2016), 1029–1039.

[13] S. Denisov, *On the growth of polynomials orthogonal on the unit circle with a weight w that satisfies $w, w^{-1} \in L_\infty(T)$*, Math. Sbornik, 209 (2018), 985–1018.

[14] S. Denisov, K. Rush, *Orthogonal polynomials on the circle for the weight w satisfying conditions $w, w^{-1} \in BMO$*, Constr. Approx. 46 (2017), no. 2, 285–303.

[15] S. Denisov, K. Rush, *On Schur parameters in Steklov's problem.* J. Approx. Theory, 215 (2017), 68–91.

[16] G. Freud, *Orthogonal Polynomials*, Pergamon Press/ Akademiai Kiado, Budapest, 1971.

[17] J.J. Guadalupe, M. Perez, F.J. Ruiz, J.L. Varona, *Asymptotic Behavior of orthogonal polynomials relative to measures with mass points,* 40(1993), 331-344.

[18] T.H. Koornwinder, *Orthogonal Polynomials with Weight Function* $(1-x)^\alpha (1+x)^\beta + M\delta(x+1) + N\delta(x-1)$, Canad. Math. Bull., 27(1984), 205-214.

[19] P. Lancaster, M. Tismenetsky, *The Theory of Matrices*, Academic Press, San Diego, 1985.

[20] Eli Levin, D.S. Lubinsky, *Orthogonal Polynomials for Exponential Weights*, Springer, New York, 2001.

[21] Eli Levin, D.S. Lubinsky, *Bounds and asymptotics for orthogonal polynomials for varying weights,* Springer Briefs in Mathematics. Springer, Cham, 2018.

[22] Eli Levin, D.S. Lubinsky, *Bounds on orthogonal polynomials and separation of their zeros,* J. Spectr. Theory, 12 (2022), 497–513.

[23] D.S. Lubinsky, *The effect of adding endpoint masspoints on bounds for orthogonal polynomials*, Dolomites Research Notes on Approximation, 14(2021), 42-55.
[24] D.S. Lubinsky, *On Zeros, Bounds, and Asymptotics for Orthogonal Polynomials on the Unit Circle,* Math. Sbornik., 213(2022), 31-49.
[25] H.N. Mhaskar, H. N. *Bounds for certain Freud-type orthogonal polynomials,* J. Approx. Theory 63 (1990), 238–254.
[26] H.N. Mhaskar, *Weighted Polynomial Approximation*, World Scientific, Singapore, 1996.
[27] E.A. Rahmanov, *Steklov's conjecture in the theory of orthogonal polynomials,* (Russian) Mat. Sb. (N.S.) 108(150) (1979), 581–608.
[28] E.A. Rakhmanov, *Estimates of the growth of orthogonal polynomials whose weight is bounded away from zero,* (Russian) Mat. Sb. (N.S.) 114(156) (1981), 269–298.
[29] E.A. Rakhmanov, *On Asymptotic Properties of Polynomials Orthogonal on the Circle with Weights not Satisfying Szegő's Condition*, Math. USSR. Sbornik, 58(1987), 149-167.
[30] B. Simon, *Orthogonal Polynomials on the Unit Circle*, Parts 1 and 2, American Mathematical Society, Providence, 2005.
[31] B. Simon, *Szegő's Theorem and its Descendants*, Princeton University Press, Princeton, 2011.

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA., lubinsky@math.gatech.edu