MORE ON DISTRIBUTION OF EIGENVALUES OF SMOOTH TOEPLITZ MATRICES

D. S. LUBINSKY

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Abstract. We obtain further results on distribution of eigenvalues of growing size Toeplitz matrices $[a_{n+k-j}]_{1\leq j,k\leq n}$ as $n\rightarrow\infty$, when the entries $\{a_j\}$ are "smooth" in the sense, for example, that for some $\alpha>0$,

$$\frac{a_{j-1}a_{j+1}}{a_j^2}=1-\frac{1}{\alpha j}\left(1+o\left(1\right)\right),\ j\rightarrow\infty.$$

In particular we consider distributions that involve absolute values of eigenvalues, or their real parts, and obtain an upper bound for the rate of decay of determinants.

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1. Introduction and Results

The distribution of eigenvalues of Toeplitz matrices is a much studied topic. Most results deal with the eigenvalues of $[c_{k-j}]_{1\leq j,k\leq n}$, where $\{c_j\}$ are the trigonometric moments of some real valued function [3]. There are generalizations to block matrices, multilevel settings and variable coefficients - see for example, [2], [5], [12], [14]. There is also an alternative setting where the entries of the Toeplitz matrix are power series coefficients. For example, Polya [9] proved that if $f\left(z\right)=\sum_{j=0}^{\infty}a_j/z^j$ can be analytically continued to a function analytic in the complex plane outside a set of logarithmic capacity $\tau\geq 0$, then

$$\limsup_{n\rightarrow\infty}\left|\det\left[a_{n-j+k}\right]_{1\leq j,k\leq n}\right|^{1/n^2}\leq\tau.$$

See [8] for a partial survey.

Toeplitz matrices also arise in Padé approximation. Let $f\left(z\right)=\sum_{j=0}^{\infty}a_jz^j$ be a formal power series, and $m,n\geq 0$. The (m,n) Padé approximant to f is a rational function $[m/n]=P/Q$ where P has degree at most m, Q has degree at most n and is not identically 0, and

$$\left(fQ-P\right)\left(z\right)=O\left(z^{m+n+1}\right),$$

in the sense that the power series on the left-hand side has 0 as the coefficient of z^j, provided $0 \leq j \leq m+n$. Q, suitably normalized, admits the representation [1]

$$Q(z) = \det \begin{bmatrix} a_m & a_{m+1} & \cdots & a_{m+n} \\ a_{m-1} & a_m & \cdots & a_{m+n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m-n+1} & a_{m-n+2} & \cdots & a_{m+1} \\ z^n & z^{n-1} & \cdots & 1 \end{bmatrix},$$

where we set $a_j = 0$ if $j < 0$, and we assume that the determinant does not vanish identically. In particular, the constant coefficient is the determinant of

$$A_{mn} = [a_{m-j+k}]_{1 \leq j,k \leq n}. \tag{1.1}$$

We let $\Lambda(B)$ denote the list of eigenvalues of a square matrix B, repeated according to multiplicity.

There is folklore that Padé approximants behave well when the coefficients are "smooth". When $a_j \neq 0$ for large enough j, the author attempted to quantify this using

$$q_j = \frac{a_{j-1}a_{j+1}}{a_j^2}. \tag{1.2}$$

In particular if n is fixed, and

$$\lim_{j \to \infty} q_j = q, \tag{1.3}$$

it was shown [6, p. 308] that

$$\lim_{m \to \infty} \det(A_{mn}) / a_m^n = \prod_{j=1}^{n-1} \left(1 - q^j\right)^{n-j}.$$

This is not useful if q is a root of unity, so additional assumptions are required for that case: if there is an asymptotic expansion, so that for each $\ell \geq 1$,

$$q_j = q - \frac{c_1}{j} + \frac{c_2}{j^2} + ... + \frac{c_\ell}{j^\ell} + O\left(j^{-\ell-1}\right),$$

where $c_1 \neq 0$, then [6, p. 309] for each fixed $n \geq 1$,

$$\lim_{m \to \infty} \det(A_{mn}) / \left\{ a_m^n \prod_{j=1}^{n-1} \left(1 - q_m^j\right)^{n-j} \right\} = 1.$$

Rusak and Starovoitov [10] showed that when n grows, but $n = o\left(m^{1/3}\right)$, this last relation persists.

The most interesting and challenging case is the "diagonal" one where $m = n \to \infty$. One situation where analysis is possible, is where (1.3) holds with $|q| < 1$, but this leads to a very narrow class of functions [7]. Probably $q = 1$ is the most interesting case. As a first step in [8], the author analyzed distribution of eigenvalues of A_{mn} when m/n remains bounded above and below, while $n \to \infty$. Singular values and other aspects were further studied in [11], see also related material in [4], [5], [13]. One class of functions to which our results are applicable is

$$f(z) = \sum_{j=0}^{\infty} z^j / (j!)^{1/\alpha}, \alpha > 0,$$

an entire function of order α. Here q_j of (1.2) satisfies

$$q_j = \exp\left(-\frac{1}{\alpha j} + O\left(\frac{1}{j^2}\right)\right).$$

This is also true for the Mittag-Leffler function

$$f(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(j/\alpha + \beta)},$$

any $\beta \in \mathbb{C}\backslash(-\infty, 0]$. For the hypergeometric function with parameters $\{c_i\}_{i=1}^k, \{d_i\}_{i=1}^\ell$ in $\mathbb{C}\backslash(-\infty, 0]$,

$$f(z) = \sum_{j=0}^{\infty} \frac{(c_1)_j (c_2)_j \dots (c_k)_j}{(d_1)_j (d_2)_j \dots (d_\ell)_j} z^j,$$

where $(c)_j = c(c+1)\dots(c+j-1)$ is the usual Pochhammer symbol, and $\ell \geq k+1$,

$$q_j = \exp\left(-\frac{(\ell - k)}{j} + O\left(\frac{1}{j^2}\right)\right).$$

In order to handle more general power series, one replaces multiples of j by more general sequences $\{\rho_j\}_{j \geq 1}$, so that one assumes

$$q_j = \exp\left(-\frac{1}{\rho_j}(1 + o(1))\right).$$

The precise technical restrictions on $\{\rho_j\}$ are given below. Using a similarity transformation, the author proved in [8] that when m/n remains bounded above and below by positive constants,

(I) As $n \to \infty$,

$$\max_{\lambda \in \Lambda(A_{mn}/a_m)} |\lambda| = \sqrt{2\pi\rho_m}(1 + o(1)). \tag{1.4}$$

(II) The set of limit points of $\left\{\Lambda\left(A_{mn}/\left(a_m\sqrt{2\pi\rho_m}\right)\right) : n \geq 1\right\}$ *is* $[0,1]$.

There were also results on distribution of eigenvalues. Define the scaled counting measure

$$\mu_{mn} = \frac{1}{n} \sum_{\lambda \in \Lambda(A_{mn}/a_m)} \delta_{\lambda/\sqrt{2\pi\rho_m}}.$$

We showed that
(III) As $n \to \infty$,

$$d\mu_{mn} \xrightarrow{*} d\delta_0$$

in the sense that for every real valued function f *defined and continuous in some open subset of the plane containing* $[0,1]$,

$$\lim_{n\to\infty} \int f \; d\mu_{mn} = f(0). \tag{1.5}$$

This says that in some sense, the eigenvalues cluster around 0. If however, we weight the eigenvalues suitably, then we obtain more interesting weak convergence results. There were two weightings used in [8]. The first was

$$\mu_{mn}^{[2]} = \frac{1}{n\sqrt{\pi\rho_m}} \sum_{\lambda \in \Lambda(A_{mn}/a_m)} \lambda^2 \delta_{\lambda/\sqrt{2\pi\rho_m}}. \tag{1.6}$$

Under suitable conditions, it was shown in [8] that
(IV) As $n \to \infty$,

$$d\mu_{mn}^{[2]} \xrightarrow{*} t\sqrt{\frac{2}{\pi\left|\log t\right|}}\; dt$$

in the sense that for every real valued function f *defined and continuous in some open subset of the plane containing* $[0,1]$,

$$\lim_{n\to\infty} \int f \; d\mu_{mn}^{[2]} = \int_0^1 f(t)\, t\sqrt{\frac{2}{\pi\left|\log t\right|}} dt. \tag{1.7}$$

Under addtional conditions on $\left\{\rho_j\right\}$, we studied the weak convergence of

$$\mu_{mn}^{[1]} = \frac{1}{n} \sum_{\lambda \in \Lambda(A_{mn}/a_m)} (\operatorname{Re}\lambda)\, \delta_{\lambda/\sqrt{2\pi\rho_m}}, \tag{1.8}$$

and proved:
(V) As $n \to \infty$,

$$d\mu_{mn}^{[1]} \xrightarrow{*} \left|\pi \log t\right|^{-1/2} dt \tag{1.9}$$

in the sense that for each function f defined and continuous in an open subset of the plane containing $[0,1]$,

$$\lim_{n\to\infty}\int f\ d\mu_{mn}^{[1]}=\int_0^1 f(t)\,|\pi\log t|^{-1/2}\,dt. \tag{1.10}$$

In particular, the assertions (I) to (V) hold for the three examples of power series listed above.

In this paper we focus on counting measures that weight absolute values of the eigenvalues, or absolute values of their real parts. We also obtain an upper bound on the determinant of A_{mn}, and pose some problems. At this stage, we present the technical conditions on $\{\rho_j\}$. In understanding them, it is good to think of $\rho_j=j$ or $\rho_j=j(\log(j+1))^{\Delta}$ for some real Δ.

Our weakest hypotheses on the comparison sequence are given in:

Definition 1.1
Let $\{\rho_j\}_{j\geq 1}$ be an increasing sequence of positive numbers, with limit ∞, with

$$\lim_{j\to\infty}\rho_j/j^2=0; \tag{1.11}$$

$$\limsup_{j\to\infty}\rho_{2j}/\rho_j<\infty; \tag{1.12}$$

and such that for each $D>0$,

$$\lim_{k\to\infty}\left(\max_{|j|\leq\sqrt{D\rho_k}}\left|1-\frac{\rho_{k+j}}{\rho_k}\right|\right)=0. \tag{1.13}$$

*Then we call $\{\rho_j\}_{j\geq 1}$ an **asymptotic comparison sequence**.*

We prove:

Theorem 1.2
Assume that $\{a_j\}$ is a non-zero sequence of complex numbers such that for some comparison sequence $\{\rho_j\}_{j\geq 1}$,

$$q_j=\frac{a_{j-1}a_{j+1}}{a_j^2}=\exp\left(-\frac{1}{\rho_j}(1+o(1))\right). \tag{1.14}$$

Let $R>1$ and for $n\geq 1$, let $m=m(n)$ satisfy

$$\frac{1}{R}\leq\frac{m}{n}\leq R.$$

(a) For $n \geq 1$, let

$$\tau_{mn} = \frac{1}{n\sqrt{\pi \rho_m}} \sum_{\lambda \in \Lambda(A_{mn}/a_m)} |\lambda|^2 \, \delta_{|\lambda|/\sqrt{2\pi\rho_m}}. \tag{1.15}$$

Then

$$\tau_{mn} \xrightarrow{*} t\sqrt{\frac{2}{\pi |\log t|}} \, dt.$$

in the sense that if $g : [0, \infty) \to \mathbb{R}$ is continuous,

$$\lim_{n\to\infty} \int g(t) \, d\tau_{mn}(t) = \int_0^1 g(t) \, t \sqrt{\frac{2}{\pi |\log t|}} \, dt. \tag{1.16}$$

(b) Let $f : (0, \infty) \to \mathbb{R}$ be continuous and nonnegative in $(0, \delta)$ for some $\delta > 0$. Then

$$\liminf_{n\to\infty} \int_{0+}^1 f(t) \, d\tau_{mn}(t) \geq \int_0^1 f(t) \, t \sqrt{\frac{2}{\pi |\log t|}} dt. \tag{1.17}$$

The integral on the left is interpreted as excluding any masspoints at 0. The integral on the right may $= \infty$.
(c) If we restrict n to those indices for which $\det(A_{mn}) \neq 0$, then

$$|\det(A_{mn})|^{1/n} = |a_m| \sqrt{2\pi\rho_m} \exp\left(-\frac{\zeta_n}{\sqrt{2\pi\rho_m}} \right), \tag{1.18}$$

where $\zeta_n \to \infty$ as $n \to \infty$.
Remarks
(a) Note that if for example $m = m(n) = n$, and the power series $f(z) = \sum_{j=0}^{\infty} a_j z^j$ is not the Maclaurin series of a rational function, then $\det(A_{nn}) \neq 0$ for infinitely many n [1].
(b) Of course it would be useful to know the rate as which $\zeta_n \to \infty$, but (d) at least gives an upper bound for the rate of decay or growth of $\det(A_{mn})$.
(c) Under additional conditions, we can replace $|\lambda|$ by $|\mathrm{Re}\,\lambda|$:

Definition 1.3
Let $\{\rho_j\}_{j\geq 1}$ be an asymptotic comparison sequence in the sense of Definition 1.1. Assume in addition, that for each $D > 0$,

$$\lim_{k\to\infty} \left(\max_{1\leq |j| \leq \sqrt{D\rho_k \log \rho_k}} \left| 1 - \frac{\rho_{k+j}}{\rho_k} \right| \frac{\rho_k^{3/4}}{j} \right) = 0, \tag{1.19}$$

and

$$\lim_{k\to\infty}\left(\max_{1\le|j|\le\sqrt{D\rho_k\log\rho_k}}\left|\frac{1}{\rho_{k+j}}+\frac{1}{\rho_{k-j}}-\frac{2}{\rho_k}\right|\frac{\rho_k^2}{|j|}\right)=0. \tag{1.20}$$

Then we call $\{\rho_j\}_{j\ge1}$ *a* ***smooth asymptotic comparison sequence.***

It is straightforward to check that $\rho_j = j$ satisfies the requirements of both definitions. Several other examples are presented in [8]. We prove:

Theorem 1.4

Assume that $\{a_j\}$ *is a non-zero sequence of complex numbers such that for some comparison sequence* $\{\rho_j\}_{j\ge1}$,

$$q_j=\frac{a_{j-1}a_{j+1}}{a_j^2}=\exp\left(-\frac{1}{\rho_j}\left(1+o\left(\rho_j^{-1/2}\right)\right)\right). \tag{1.21}$$

Let $R>1$ *and for* $n\ge1$, *let* $m=m(n)$ *satisfy*

$$\frac{1}{R}\le\frac{m}{n}\le R.$$

For $n\ge1$, let

$$\omega_{mn}=\frac{1}{n}\sum_{\lambda\in\Lambda(A_{mn}/a_m)}|\operatorname{Re}\lambda|\,\delta_{|\operatorname{Re}\lambda|/\sqrt{2\pi\rho_m}}.$$

(a) Then

$$d\omega_{mn}\xrightarrow{*}|\pi\log t|^{-1/2}\,dt.$$

in the sense that if $g:[0,\infty)\to\mathbb{R}$ *is continuous,*

$$\lim_{n\to\infty}\int g(t)\,d\omega_{mn}(t)=\int_0^1 g(t)\,|\pi\log t|^{-1/2}\,dt. \tag{1.22}$$

(b) Let $f:(0,\infty)\to\mathbb{R}$ *be continuous and nonnegative in* $(0,\delta)$ *for some* $\delta>0$. *Then*

$$\liminf_{n\to\infty}\int_{0+}^1 f(t)\,d\omega_{mn}(t)\ge\int_0^1 f(t)\,|\pi\log t|^{-1/2}\,dt. \tag{1.23}$$

The integral on the left is interpreted as excluding any masspoints at 0. *The integral on the right may* $=\infty$.

(c) For $\varepsilon\in(0,1)$,

(1.24)

$$\lim_{n\to\infty}\frac{\sqrt{2\pi\rho_m}}{n}\log\left(\prod_{\lambda\in\Lambda(A_{mn}/a_m),\ \frac{|\operatorname{Re}\lambda|}{\sqrt{2\pi\rho_m}}\ge\varepsilon}\frac{\sqrt{2\pi\rho_m}}{|\operatorname{Re}\lambda|}\right)=\int_\varepsilon^1\frac{1}{t}\left|\pi^{-1}\log t\right|^{1/2}dt.$$

Moreover,

$$\lim_{n\to\infty}\frac{\sqrt{2\pi\rho_m}}{n}\log\left(\prod_{\lambda\in\Lambda(A_{mn}/a_m),\lambda\neq 0}\frac{\sqrt{2\pi\rho_m}}{|\mathrm{Re}\,\lambda|}\right)=\infty. \tag{1.25}$$

In terms of applications to Padé approximation, it would be very useful to have estimates for the full $\det(A_{mn})$. This would of course be possible if we can handle the eigenvalues close to 0. This suggests

Problem 1.5
Formulate conditions for $\det(A_{mn})$ *to be non-zero.*
A more quantitative goal would be:

Problem 1.6
Estimate below the eigenvalue of smallest modulus of A_{mn}*, under suitable conditions.*
The most ambitious would be:

Problem 1.7
Formulate conditions that permit some form of asymptotics for $\det(A_{mn})$.
This paper is organised as follows: in Section 2, we recall some results from [8]. In Section 3, we prove the results. Throughout, $C, C_1, C_2, ...$ denote constants independent of n, x, z, t and possibly other specified parameters. The same symbol does not necessarily denote the same constant in different occurrences.

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Its is a privilege to dedicate this article to the 80th birthday of Dany Leviatan. Smoothness plays a role in a lot of his research on polynomial approximation, but of course the smoothness we discuss here is quite different.

2. Preliminaries

In this section, we recall a number of results from [8].

Theorem 2.1
Assume the hypotheses of Theorem 1.2. Fix $k\geq 1$*. Then as* $n\to\infty$,

$$Tr\left[\left(\frac{A_{mn}}{a_m}\right)^k\right]=n\,(2\pi\rho_m)^{(k-1)/2}\,\frac{1+o(1)}{\sqrt{k}}. \tag{2.1}$$

Proof

See Theorem 3.1 in [8, p. 348]. ■

Lemma 2.2
(a) Assume the hypotheses of Theorem 1.2. Then

$$\max_{\lambda\in\Lambda(A_{mn}/a_m)} |\lambda| = \sqrt{2\pi\rho_m}\,(1+o(1)) \tag{2.2}$$

and

$$\frac{1}{n\sqrt{\rho_m}} \sum_{\lambda\in\Lambda(A_{mn}/a_m)} |\mathrm{Im}\,\lambda|^2 = o(1)\,. \tag{2.3}$$

(b) Assume the hypotheses of Theorem 1.4. Then

$$\min_{\lambda\in\Lambda(A_{mn}/a_m)} \mathrm{Re}\,(\lambda) \geq -o(1) \tag{2.4}$$

and

$$\frac{1}{n} \sum_{\lambda\in\Lambda(A_{mn}/a_m)} |\mathrm{Re}\,\lambda| = 1+o(1)\,. \tag{2.5}$$

Proof
(a) These are Theorem 1.2(I) in [8, p. 337] and Lemma 4.2(a) in [8, p. 354].
(b) See Lemma 6.1(a) in [8, p. 361] and Theorem 1.4(II) in [8, p. 338]. ■

3. **Proof of Theorems 1.2 and 1.4**

We begin by replacing λ by $\mathrm{Re}\,\lambda$ or $|\lambda|$ in the eigenvalue counting measures.

Lemma 3.1
(a) Assume the hypotheses of Theorem 1.4. Then for $k \geq 1$,

$$\lim_{n\to\infty} \frac{\sqrt{2\pi\rho_m}}{n} \sum_{\lambda\in\Lambda(A_{mn}/a_m)} \left|\frac{\mathrm{Re}\,\lambda}{\sqrt{2\pi\rho_m}}\right|^k = \frac{1}{\sqrt{k}}.$$

(b) Assume the hypotheses of Theorem 1.2. Then for $k \geq 2$,

$$\lim_{n\to\infty} \frac{\sqrt{2\pi\rho_m}}{n} \sum_{\lambda\in\Lambda(A_{mn}/a_m)} \left|\frac{\lambda}{\sqrt{2\pi\rho_m}}\right|^k = \frac{1}{\sqrt{k}}.$$

Proof
(a) First let $k \geq 2$. Observe that if $z = x + iy$, then

$$\begin{aligned} \operatorname{Re}\left(z^k\right) - x^k &= \operatorname{Re}\left(\sum_{j=1}^{k} \binom{k}{j} (iy)^j \, x^{k-j}\right) \\ &= \sum_{j=2, j \text{ even}}^{k} \binom{k}{j} (-1)^{j/2} \, y^j x^{k-j}, \end{aligned}$$

so that for some constant C depending only on k,

$$\left|\operatorname{Re}\left(z^k\right) - x^k\right| \leq C y^2 \left|z\right|^{k-2}.$$

Using (2.2), and this last inequality,

$$\begin{aligned} &\left|\frac{1}{n} \operatorname{Re}\left(Tr\left[\left(\frac{A_{mn}}{a_m}\right)^k\right]\right) - \frac{1}{n} \sum_{\lambda \in \Lambda(A_{mn}/a_m)} (\operatorname{Re}\lambda)^k\right| \\ (3.1) \qquad &\leq C \left(\sqrt{\rho_m}\right)^{k-2} \frac{1}{n} \sum_{\lambda \in \Lambda(A_{mn}/a_m)} \left|\operatorname{Im}\lambda\right|^2 = o\left(\sqrt{\rho_m}\right)^{k-1}, \end{aligned}$$

by (2.3). Next, let $\varepsilon \in (0, 1)$. From (2.4), there exists N such that for $n \geq N$, and for all $\lambda \in \Lambda\left(A_{mn}/a_m\right)$, $\operatorname{Re}(\lambda) \geq -\varepsilon$. Then

$$0 \leq \frac{1}{n} \sum_{\lambda \in \Lambda(A_{mn}/a_m)} \left(\left|\operatorname{Re}\lambda\right|^k - (\operatorname{Re}\lambda)^k\right) \leq \frac{2}{n} \sum_{\lambda \in \Lambda(A_{mn}/a_m), \operatorname{Re}\lambda < 0} \left|\operatorname{Re}\lambda\right|^k \leq 2\varepsilon.$$

Hence

$$\left|\frac{1}{n} \operatorname{Re}\left(Tr\left[\left(\frac{A_{mn}}{a_m}\right)^k\right]\right) - \frac{1}{n} \sum_{\lambda \in \Lambda(A_{mn}/a_m)} \left|\operatorname{Re}\lambda\right|^k\right| \leq 2\varepsilon + o\left(\sqrt{\rho_m}\right)^{k-1}.$$

Then Theorem 2.1 gives the result for $k \geq 2$. If $k = 1$, the result follows from (2.5).
(b) Let $k \geq 2$. We first show there exists $C > 0$ such that for $z = x + iy$,

$$(3.2) \qquad \left|\left|z\right|^k - \operatorname{Re}\left(z^k\right)\right| \leq C \left|z\right|^{k-2} y^2.$$

If first $|x| \leq 2|y|$, then the left-hand side of (3.2) is bounded above by $C|y|^k$, and hence by $C|z|^{k-2}y^2$. Now suppose that $|x| > 2|y|$. Then

$$\begin{aligned} & \left| |z|^k - \operatorname{Re}\left(z^k\right) \right| \\ = \ & |x|^k \left| \left(1 + \left(\frac{y}{x}\right)^2\right)^{k/2} - 1 - \sum_{j=2, j \text{ even}}^{k} \binom{k}{j} (-1)^{j/2} \left(\frac{y}{x}\right)^j \right| \\ \leq \ & C|x|^k \left(\frac{y}{x}\right)^2 \leq C|z|^{k-2} y^2. \end{aligned}$$

So we have (3.2) in all cases. Then by (2.5),

$$\begin{aligned} & \left| \frac{1}{n} \sum_{\lambda \in \Lambda(A_{mn}/a_m)} |\lambda|^k - \frac{1}{n} \operatorname{Re}\left(Tr\left[\left(\frac{A_{mn}}{a_m} \right)^k \right] \right) \right| \\ \leq \ & C\left(\sqrt{\rho_m}\right)^{k-2} \frac{1}{n} \sum_{\lambda \in \Lambda(A_{mn}/a_m)} |\operatorname{Im} \lambda|^2 = o\left(\sqrt{\rho_m}\right)^{k-1}, \end{aligned}$$

by (2.3). Again Theorem 2.1 gives the result. ■

Proof of Theorem 1.4
Recall that

$$\omega_{mn} = \frac{1}{n} \sum_{\lambda \in \Lambda(A_{mn}/a_m)} |\operatorname{Re} \lambda| \, \delta_{|\operatorname{Re}\lambda|/\sqrt{2\pi\rho_m}}.$$

(a) Now by Lemma 3.1(a), for $j \geq 0$,

$$\begin{aligned} \lim_{n\to\infty} \int t^j d\omega_{mn}(t) &= \frac{1}{n} \sum_{\lambda \in \Lambda(A_{mn}/a_m)} |\operatorname{Re}\lambda| \left(\frac{|\operatorname{Re}\lambda|}{\sqrt{2\pi\rho_m}} \right)^j \\ &= \frac{\sqrt{2\pi\rho_m}}{n} \sum_{\lambda \in \Lambda(A_{mn}/a_m)} \left(\frac{|\operatorname{Re}\lambda|}{\sqrt{2\pi\rho_m}} \right)^{j+1} \\ &= \frac{1}{\sqrt{j+1}} (1 + o(1)). \end{aligned}$$

The latter are the moments of the probability measure $|\pi \log t|^{-1/2} dt$ on $[0, 1]$. See the proof of Lemma 6.2 in [8, p. 364]. Hence

$$\lim_{n\to\infty} \int t^j d\omega_{mn}(t) = \int_0^1 t^j |\pi \log t|^{-1/2} dt.$$

As all $\{\omega_{mn}\}$ have support contained in a fixed compact set, (as follows from (2.2)), the weak convergence in (1.22) follows.

(b) Let $\varepsilon \in (0,1)$ and

$$h(t) = \begin{cases} 0, & 0 \leq t < \varepsilon \\ f(t), & t \in [\varepsilon, \infty) \end{cases}.$$

We assume $f(\varepsilon) > 0$. Since for large enough n, ω_{mn} has support in $\left[0, \frac{3}{2}\right]$, we may assume that f vanishes in $[2, \infty)$. Then h is upper semi-continuous in $[0,2]$, so there exists a decreasing sequence of continuous functions $\{g_k\}$ on $[0,2]$ with pointwise limit h. We may extend each g_k to $[0,\infty)$ by defining it to be constant in $[2,\infty)$. Then for n large enough so that the support of ω_{mn} lies in $[0,2]$,

$$\limsup_{n\to\infty} \int h(t)\, d\omega_{mn}(t) \leq \limsup_{n\to\infty} \int g_k(t)\, d\omega_{mn}(t) = \int_0^1 g_k(t) \left|\pi \log t\right|^{-1/2},$$

by (a). Letting $k \to \infty$, gives, via the Monotone Convergence Theorem,

$$\limsup_{n\to\infty} \int h(t)\, d\omega_{mn}(t) \leq \int_0^1 h(t) \left|\pi \log t\right|^{-1/2} dt.$$

Similarly if we redefine $h(\varepsilon) = 0$, giving h_-we obtain a lower semi-continuous function h_-. Proceeding as above, we obtain

$$\liminf_{n\to\infty} \int h_-(t)\, d\omega_{mn}(t) \geq \int_0^1 h_-(t) \left|\pi \log t\right|^{-1/2} dt.$$

Next, from the definition of ω_{mn},

$$\begin{aligned} 0 &\leq f(\varepsilon)\, \omega_{mn}(\{\varepsilon\}) \leq f(\varepsilon) \max_{\lambda \in \Lambda(Amn)/a_m} |\lambda| / n \\ &\leq (1 + o(1))\, f(\varepsilon) \sqrt{2\pi\rho_m}/n = o(1), \end{aligned}$$

by (1.4) and (1.11). We deduce from all the above that

$$\lim_{n\to\infty} \int_\varepsilon^1 f(t)\, d\omega_{mn}(t) = \int_\varepsilon^1 f(t) \left|\pi \log t\right|^{-1/2} dt. \tag{3.3}$$

We deduce that for each $\varepsilon > 0$,

$$\liminf_{n\to\infty} \int_{0+}^1 f(t)\, d\omega_{mn}(t) \geq \int_\varepsilon^1 f(t) \left|\pi \log t\right|^{-1/2} dt.$$

Now let $\varepsilon \to 0+$ to obtain (1.23).

(c) Choose

$$f(t) = t^{-1} \log \frac{1}{t}, t \in (0, \infty).$$

Let $\varepsilon \in (0,1)$. We obtain from (b),

$$\lim_{n\to\infty} \frac{\sqrt{2\pi\rho_m}}{n} \sum_{\lambda \in \Lambda(A_{mn}/a_m),\, \frac{|\mathrm{Re}\,\lambda|}{\sqrt{2\pi\rho_m}} \geq \varepsilon} \log \left| \frac{\sqrt{2\pi\rho_m}}{\mathrm{Re}\,\lambda} \right| = \pi^{-1/2} \int_\varepsilon^1 t^{-1} \left|\log t\right|^{1/2} dt.$$

That is, (1.24) holds. Then also, as $\log\left|\frac{\sqrt{2\pi\rho_m}}{\operatorname{Re}\lambda}\right|>0$ if $\frac{|\operatorname{Re}\lambda|}{\sqrt{2\pi\rho_m}}<\varepsilon$,

$$\liminf_{n\to\infty}\frac{\sqrt{2\pi\rho_m}}{n}\sum_{\lambda\in\Lambda(A_{mn}/a_m),|\operatorname{Re}\lambda|>0}\log\left|\frac{\sqrt{2\pi\rho_m}}{\operatorname{Re}\lambda}\right|\geq\pi^{-1/2}\int_\varepsilon^1 t^{-1}\left|\log t\right|^{1/2}dt.$$

Now let $\varepsilon\to 0$ and use divergence of the integral. ■

Proof of Theorem 1.2
Recall that

$$\tau_{mn}=\frac{1}{n\sqrt{\pi\rho_m}}\sum_{\lambda\in\Lambda(A_{mn}/a_m)}|\lambda|^2\,\delta_{|\lambda|/\sqrt{2\pi\rho_m}}.$$

(a) We have for $j\geq 0$,

$$\begin{aligned}\int t^j d\tau_{mn} &= \frac{1}{n\sqrt{\pi\rho_m}}\sum_{\lambda\in\Lambda(A_{mn}/a_m)}|\lambda|^2\left(\frac{|\lambda|}{\sqrt{2\pi\rho_m}}\right)^j\\ &= \frac{\sqrt{2}\sqrt{2\pi\rho_m}}{n}\sum_{\lambda\in\Lambda(A_{mn}/a_m)}\left(\frac{|\lambda|}{\sqrt{2\pi\rho_m}}\right)^{j+2}\\ &= \sqrt{\frac{2}{j+2}}\left(1+o\left(1\right)\right),\end{aligned}$$

by Lemma 3.1(b). As shown in [8, p. 360, Proof of Lemma 5.2], the unique probability distribution with these moments is $t\sqrt{\frac{2}{\pi|\log t|}}\,dt$. So

$$\lim_{n\to\infty}\int t^j d\tau_{mn}=\int_0^1 t^j t\sqrt{\frac{2}{\pi\left|\log t\right|}}dt.$$

As the supports of all $\{\tau_{mn}\}$ are contained in a compact set, because of (2.2), the weak convergence follows.
(b) This is the same as in the proof of Theorem 1.4(b).
(c) Choose

$$f\left(t\right)=t^{-2}\log\frac{1}{t},t\in\left(0,\infty\right).$$

We obtain from (b),

$$\lim_{n\to\infty}\frac{\sqrt{2\pi\rho_m}}{n}\sum_{\lambda\in\Lambda(A_{mn}/a_m),\frac{|\lambda|}{\sqrt{2\pi\rho_m}}\geq\varepsilon}\log\left|\frac{\sqrt{2\pi\rho_m}}{\lambda}\right|=\pi^{-1/2}\int_\varepsilon^1 t^{-1}\left|\log t\right|^{1/2}dt.$$

Hence also

$$\liminf_{n\to\infty} \frac{\sqrt{2\pi\rho_m}}{n} \sum_{\lambda\in\Lambda(A_{mn}/a_m),\lambda\neq 0} \log\left|\frac{\sqrt{2\pi\rho_m}}{\lambda}\right| \geq \pi^{-1/2}\int_\varepsilon^1 t^{-1}\left|\log t\right|^{1/2} dt$$

as terms excluded all have $\frac{\sqrt{2\pi\rho_m}}{|\lambda|} > \varepsilon^{-1}$. Since

$$\int_0^1 t^{-1}\left|\log t\right|^{1/2} dt = \infty,$$

it follows that

$$\frac{\sqrt{2\pi\rho_m}}{n} \sum_{\lambda\in\Lambda(A_{mn}/a_m),\lambda\neq 0} \log\left|\frac{\sqrt{2\pi\rho_m}}{\lambda}\right| = \zeta_n,$$

where $\zeta_n \to \infty$ as $n \to \infty$. If $\det(A_{mn}) \neq 0$, this gives

$$\frac{\sqrt{2\pi\rho_m}}{n}\log\frac{\left(\sqrt{2\pi\rho_m}\right)^n}{\left|\det\left(A_{mn}\right)/a_m^n\right|} = \zeta_n$$

$$\Rightarrow \sqrt{2\pi\rho_m}\log\frac{\sqrt{2\pi\rho_m}}{\left|\det\left(A_{mn}\right)\right|^{1/n}/\left|a_m\right|} = \zeta_n$$

$$\Rightarrow \left|\det\left(A_{mn}\right)\right|^{1/n} = \left|a_m\right|\sqrt{2\pi\rho_m}\exp\left(-\frac{\zeta_n}{\sqrt{2\pi\rho_m}}\right).$$

■

References

[1] G.A. Baker, P. Graves-Morris, *Padé Approximants*, 2nd Edition, Cambridge University Press, Cambridge, 1996.

[2] C. Garoni, S. Serra-Capizzano, *Generalized Locally Toeplitz Sequences: Theory and Applications*, Vol. 1, Springer, Cham., 2017.

[3] U. Grenander, G. Szegő, *Toeplitz Forms*, Chelsea, New York, 1958.

[4] L. Golinskii, S. Serra-Capizzano, *The asymptotic properties of the spectrum of nonsymmetrically perturbed Jacobi matrix sequences,* J. Approx. Th., 144(2007), 84-102.

[5] A. Kuijlaars, S. Serra-Capizzano, *Asymptotic zero distribution of orthogonal polynomials with discontinuously varying recurrence coefficients*, J. Approx. Th., 113(2001), 142-155.

[6] D.S. Lubinsky, *Uniform Convergence of Rows of the Padé Table for Functions with Smooth Maclaurin Series Coefficients,* Constr. Approx., 3(1987), 307-330.

[7] D.S. Lubinsky, *Padé Tables of Entire Functions of Very Slow and Smooth Growth II*, Constr. Approx., 4(1988), 321-339.

[8] D.S. Lubinsky, *Distribution of Eigenvalues of Toeplitz Matrices with Smooth Entries*, Lin. Alg and its Applns., 633(2022), 332-365.

[9] G. Polya, *Über gewisse notwendige Determinantenkriterein fur die Forsetzbarkeit einer Potenzreihe*, Math. Annalen, 99(1928), 687-706.

[10] V. N. Rusak and A. P. Starovoitov, *Padé approximants for entire functions with regularly decreasing Taylor coefficients,* Mathematics Sbornik 193(2002), 63–92 (2002).

[11] A.J. A Schiavoni and S. Serra-Capizzano, *Distribution results for a special class of matrix sequences: joining approximation theory and asymptotic linear algebra*, Electron. Trans. Numer. Anal.59(2023), 1–8.

[12] P. Tilli, *A note on the spectral distribution of Toeplitz matrices,* Linear and Multilinear Algebra, 45(1998), 147-159.

[13] W.F. Trench, *A note on asymptotic zero distribution of orthogonal polynomials*, Linear Alg. Applns., 375(2003), 275-281.

[14] E.E. Tyrtyshnikov, N.L. Zamarashkin, *Spectra of multilevel Toeplitz matrices: advanced theory via simple matrix relationships*, Lin. Alg. Applns., 270(1998), 15-27.

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160, USA., lubinsky@math.gatech.edu