ZEROS OF RANDOM MÜNTZ POLYNOMIALS

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Abstract. We study the expected number of positive zeros of Müntz polynomials with real i.i.d. coefficients. For the standard Gaussian coefficients, we establish asymptotic results for the expected number of positive zeros when the exponents of Müntz mononomials that span our random Müntz polynomials have polynomial and logarithmic growth. We also present many bounds on the expected number of zeros of random Müntz polynomials with various real i.i.d. coefficients, including the case of arbitrary nontrivial real i.i.d. coefficients.

1. Introduction

Zeros of polynomials with random coefficients have been intensively studied for almost one hundred years, beginning with the paper of Bloch and Pólya [1]. We refer to the book [3], [12] and the survey [6] for information on early results. The literature on this subject increased exponentially in the past three decades, so that we mention a very incomplete list of recent papers such as [28], [20], [4], [21], [5], etc. Perhaps the most popular set of questions in this field are related to the expected number of real zeros of random polynomials, where an important role is played by the Kac-Rice integral formula, see [13, 14]. This paper is devoted to the same topic, but in a more general context of random *Müntz polynomials* that are spanned by monomials with arbitrary real exponents. Müntz polynomials were used in classical analysis and approximation theory beginning with [19] and [27]. For a sequence of distinct and non-negative real numbers $\{\lambda_k\}_{k=1}^{\infty}$, we consider random Müntz polynomials

$$P_n(x) = \sum_{k=0}^{n} c_k x^{\lambda_k}, \quad n \in \mathbb{N}, \tag{1.1}$$

with $\lambda_0 = 0$ and real i.i.d. coefficients $\{c_k\}_{k=0}^{\infty}$, and study the expected number of their real zeros $\mathbb{E}\left[N_n\left([a,b]\right)\right]$ in an interval $[a,b] \subset \mathbb{R}$. It is standard in the literature on random polynomials, especially based on the Kac-Rice integral formula, to count every real zero only once, disregarding multiplicities. We also follow this convention. Throughout this paper, the i.i.d. random variables $\{c_k\}_{k=0}^{\infty}$ are assumed non-trivial, i.e., $\mathbb{P}(c_k = 0) = q < 1$. Since $\lambda_k \in \mathbb{R}$, we study the expected number of zeros of such polynomials on $[0,\infty)$, and especially on $[0,1]$, under various conditions on the exponents $\{\lambda_k\}_{k=1}^{\infty}$ and on the random coefficients $\{c_k\}_{k=0}^{\infty}$.

In the case when the $\{\lambda_k\}_{k=1}^{\infty}$ are a sparse set of integers and the $\{c_k\}_{k=0}^{\infty}$ are i.i.d. standard Gaussian random variables, some useful estimates were established in [10], see also [2] for earlier results in this direction. In particular, it was proved in [10] that for arbitrary integers

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$\{\lambda_j\}_{j=0}^{n-1}$, we have

$$\mathbb{E}\left[N_n\left([0,1]\right)\right] \leq \frac{2}{\pi}\sqrt{n-1}$$

and moreover that for $n \geq 4$ and suitable $\{\lambda_j\}_{j=0}^{n-1}$, we have

$$\mathbb{E}\left[N_n\left([0,1]\right)\right] \geq \frac{\pi-\sqrt{3}}{16\pi}\sqrt{n-2}+\frac{1}{7}.$$

We significantly improve and generalize the above results by providing asymptotics for $\mathbb{E}\left[N_n\left([0,1]\right)\right]$ when the coefficients $\{c_k\}_{k=0}^{\infty}$ are i.i.d. standard Gaussian, and the exponents $\{\lambda_k\}_{k=1}^{\infty}$ of Müntz polynomials have polynomial and logarithmic growth. These results, obtained via the Kac-Rice formula, are presented in Section 2. Section 3 contains many bounds for the expected number of zeros of random Müntz polynomials. In particular, we prove the upper bound of the form $\mathbb{E}\left[N_n\left([0,\infty)\right)\right] \leq C\sqrt{n}$ for all random Müntz polynomials with very general real i.i.d. random coefficients. It is worth pointing out that our results include sparse (or lacunary) random polynomials as a special case. All proofs for Section 2 are given in Section 5, while proofs for Sections 3 and 4 are given in Sections 6 and 7 correspondingly.

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2. Asymptotic results based on the Kac-Rice formula

Throughout this section, we assume that our random Müntz polynomials (1.1) have i.i.d. standard Gaussian coefficients. It is known that the Kac-Rice formula is valid for this kind of random polynomial, see Proposition 1.1 in [18], so that we use the formula to find the asymptotic results below. Define the kernel

$$K_n\left(x,y\right)=\sum_{j=0}^{n}\left(xy\right)^{\lambda_j},\ x,y\geq 0,$$

and for non-negative integers j,k,

$$K_n^{(j,k)}\left(x,y\right)=\frac{\partial^{j+k}}{\partial x^j\partial y^k}K_n\left(x,y\right).$$

The Kac-Rice formula states that the density of real zeros is given by

$$I_n\left(x\right)=\sqrt{\frac{K_n^{(1,1)}\left(x,x\right)K_n\left(x,x\right)-K_n^{(1,0)}\left(x,x\right)^2}{K_n\left(x,x\right)^2}}, \tag{2.1}$$

so that the expected number of zeros over an interval $[a,b]\subset[0,\infty)$ is

$$\mathbb{E}\left[N_n\left([a,b]\right)\right]=\frac{1}{\pi}\int_a^b I_n(x)\,dx, \tag{2.2}$$

We establish asymptotics for $\mathbb{E}\left[N_n\left([0,1]\right)\right]$ when the exponents are of polynomial and logarithmic growth. The Kac-Rice formula also allows us to find bounds for $\mathbb{E}\left[N_n\left([0,1]\right)\right]$ when the exponents are growing geometrically, as well as to present a general estimate for $I_n\left(x\right)$. In our first three results, we shall assume that

$$\lambda_j=\phi\left(j\right),\quad j\geq 0, \tag{2.3}$$

where $\phi : [0,\infty) \to [0,\infty)$ is continuous and strictly increasing in $[0,\infty)$, with $\phi(0) = 0$, and ϕ' is continuous and positive in $(0,\infty)$. We shall let $\phi^{[-1]}$ denote its inverse function and

$$\Psi(r) = \phi'\left(\phi^{[-1]}(r)\right), \quad r > 0. \tag{2.4}$$

First we deal with the exponents $\{\lambda_j\}_{j=0}^{\infty}$ that are of polynomial growth.

Theorem 2.1. *Assume that*

$$\lim_{r\to\infty} \Psi(r)/r = 0. \tag{2.5}$$

Let $\Delta \in \mathbb{R}$ and assume that for each fixed $y > 0$,

$$\lim_{r\to\infty} \frac{\Psi(r)}{\Psi(yr)} = y^{\Delta}. \tag{2.6}$$

Assume moreover, that there is a measurable function $g(y)$ and r_0 such that for $r \geq r_0$ and $y > 0$,

$$\frac{\Psi(r)}{\Psi(yr)} \leq g(y) \tag{2.7}$$

while

$$\int_0^{\infty} e^{-2y}\left(1+y^2\right) g(y)\, dy < \infty. \tag{2.8}$$

Then as $n \to \infty$,

$$\mathbb{E}\left[N_n([0,1])\right] = \frac{d_0}{\pi}(1+o(1)) \log \phi(n), \tag{2.9}$$

where

$$d_0 = \frac{1}{2}\sqrt{\frac{\Gamma(\Delta+3)\Gamma(\Delta+1) - \Gamma(\Delta+2)^2}{\Gamma(\Delta+1)^2}}. \tag{2.10}$$

Remark
The conditions on Ψ are similar to those defining regularly varying functions.

Examples
(I) Let

$$\phi(t) = t^{\beta}, \beta > 0.$$

The above hypotheses are satisfied with

$$\Delta = -1 + \frac{1}{\beta}.$$

so

$$d_0 = \frac{1}{2}\sqrt{\frac{\Gamma\left(2+\frac{1}{\beta}\right)\Gamma\left(\frac{1}{\beta}\right) - \Gamma\left(1+\frac{1}{\beta}\right)^2}{\Gamma\left(\frac{1}{\beta}\right)^2}}. \tag{2.11}$$

The above result becomes

$$\mathbb{E}\left[N_n([0,1])\right] = \frac{d_0\beta}{\pi}(1+o(1)) \log n. \tag{2.12}$$

In the ordinary polynomial case, where $\beta = 1$, we obtain $d_0 = \frac{1}{2}$, so

$$\mathbb{E}\left[N_n\left([0,1]\right)\right] = \frac{1}{2\pi}(1 + o(1)) \log n.$$

This agrees with the classical Kac result, where

$$4\mathbb{E}\left[N_n\left([0,1]\right)\right] = \mathbb{E}\left[N_n\left(\mathbb{R}\right)\right] = \frac{2}{\pi}(1 + o(1)) \log n.$$

(II) Let $\beta > 0, \gamma > 0$, and

$$\phi\left(t\right) = t^{\beta}\left(\log\left(t+2\right)\right)^{\gamma}, \ t \geq 0.$$

Here we still have $\Delta = -1 + 1/\beta$ and also have (2.11)-(2.12). See Section 5 for a full deduction from Theorem 2.1.

Next we consider logarithmic growth:

Theorem 2.2. *Let*

$$\lambda_n = \log(n+1), \ n \geq 0.$$

Then

$$\mathbb{E}\left[N_n\left([0,1]\right)\right] = \frac{1}{\pi}(1 + o(1)) \log\log(n+1).$$

Throughout the paper $C, C_1, C_2, \ldots$ denote positive constants that are independent of n, x, t. The same symbol may be used to indicate different constants in different occurrences.

For the geometric growth of exponents, we give two-sided bounds:

Theorem 2.3. *Let $b > 1$ and*

$$\lambda_n = b^n - 1, n \geq 0.$$

Then for some $C_1, C_2 > 0$,

$$C_1 \leq \mathbb{E}\left[N_n\left([0,1]\right)\right]/\sqrt{n} \leq C_2.$$

The Kac-Rice formula also gives the following general bound that holds for rather arbitrary real exponents $\{\lambda_k\}_{k=1}^{\infty}$.

Theorem 2.4. *For any sequence of distinct and non-negative real numbers $\{\lambda_k\}_{k=1}^{\infty}$, satisfying*

$$\liminf_{k\to\infty} \frac{\lambda_k}{\log k} = a > 0, \tag{2.13}$$

there is $C > 0$ such that

$$\mathbb{E}[N_n([0,1])] \leq C\sqrt{\log n}\sqrt{\log \sum_{k=1}^{n} \lambda_k}, \quad n \geq 2.$$

This implies, in particular, that for exponents of at most polynomial growth, the expected number of real zeros is of the order $\log n$.

Corollary 2.5. *If $\{\lambda_k\}_{k=1}^{\infty} \subset \mathbb{N}$ and $\lambda_k = O(k^p)$, where $p \geq 1$, then $\mathbb{E}[N_n(\mathbb{R})] = O(\log n)$.*

Finally, we present some general estimates for the density of real zeros:

Theorem 2.6. (a) *Let $0 < \rho < 1$. There exists $C_1 > 0$ such that for all distinct positive $\{\lambda_k\}_{k=1}^{\infty}$, all $n \geq 2$, and all $x \in (0, \rho]$,*

$$I_n(x) \leq \frac{C_1}{x} \log n. \tag{2.14}$$

(b) *Let $R > 1$. There exists $C_2 > 0$ such that for all distinct positive $\{\lambda_k\}_{k=1}^{\infty}$, all $n \geq 2$, and all $x \in [R, \infty)$,*

$$I_n(x) \leq \frac{C_2}{x} \log n. \tag{2.15}$$

(c) *For all distinct positive $\{\lambda_k\}_{k=1}^{\infty}$ and all $x \in (0, \infty)$,*

$$I_n(x) \leq \frac{\max_{1 \leq k \leq n} \lambda_k}{x}.$$

In particular, if the sequence $\{\lambda_n\}_{n=1}^{\infty}$ is bounded, so is $\{I_n(x)\}_{n=1}^{\infty}$.

Remark. The upper bound $\log n$ for $I_n(x)$ is best possible. For $\lambda_j = \log(j+1)$, $j \geq 0$, and $x = e^{-1/2}$, it is shown in Section 5.3 that $I_n(x) \sim \log n$.

3. General bounds for the expected number of zeros

While the Kac-Rice formula (2.2) proved to be a very important tool for the study of real zeros of random polynomials, it is only suitable for the case of Gaussian coefficients. Moreover, even its generalizations are much more difficult to apply if the coefficients are not Gaussian. In this section, we use a completely different approach, based on the counting of sign changes in sums of coefficients, and provide upper bounds for the expected number of zeros of Müntz polynomials with arbitrary exponents and very general random coefficients.

Our main deterministic tool for bounding the number of real roots from above is the following rule of signs due to Laguerre [15, p. 9], see also Remark 10.4.5 in [23, p. 333]. Below, we use $V(t_1, \ldots, t_k)$ to denote the number of sign changes in the sequence of real numbers $\{t_i\}_{i=1}^k$.

Theorem 3.1 (Laguerre). *For any polynomial $f(x) = \sum_{k=0}^n a_k x^k$ with real coefficients, define its partial sums by $f_k(x) = \sum_{i=0}^k a_i x^i$, $k = 0, \ldots, n$. If $b > 0$ and $f(b) \neq 0$, then the number of real zeros of f contained in the interval $(0, b]$ does not exceed $V(f_0(b), \ldots, f_n(b))$.*

Applying Laguerre's rule of signs with $b = 1$ to polynomials of the form (1.1), we immediately obtain an important corollary.

Corollary 3.2. *Let P_n be a deterministic polynomial with real coefficients of the form (1.1), i.e., we assume that $\{\lambda_k\}_{k=1}^n \subset \mathbb{N}$ is an increasing sequence. Define the sums $s_k := \sum_{i=0}^k c_i$, $k = 0, \ldots, n$. If $s_n \neq 0$ then the number of zeros of P_n in $(0, 1]$ does not exceed $V(s_0, \ldots, s_n)$.*

In fact, the above results count zeros even with multiplicities, so that the same upper bounds obviously hold if we count without multiplicities. Corollary 3.2 is very useful for our purpose because sums of random variables represent a central topic in probability theory. We now recall a result due to Erdős and Hunt, see Theorem 1 in [7].

Theorem 3.3 (Erdős and Hunt). *If $\{c_k\}_{k=0}^\infty$ are independent real random variables with the same symmetric and continuous distribution, then*

$$\mathbb{E}[V(s_0,\ldots,s_n)] \le \sum_{k=1}^n \frac{[k/2]+1}{k+1}\binom{k}{[k/2]}2^{-k} \tag{3.1}$$

where $s_k := \sum_{i=0}^k c_i,\ k=0,\ldots,n.$

This result immediately implies upper bounds for the expected number of real zeros. We first state them for regular polynomials with natural exponents on the real line, and then deduce the corresponding results for Müntz polynomials with arbitrary real exponents on the positive semi-axis by using an approximation argument.

Theorem 3.4. *If $\{c_k\}_{k=0}^\infty$ are independent real random variables with the same symmetric and continuous distribution, and $\{\lambda_k\}_{k=1}^\infty \subset \mathbb{N}$, then*

$$\mathbb{E}[N_n([0,1])] \le \sum_{k=1}^n \frac{[k/2]+1}{k+1}\binom{k}{[k/2]}2^{-k} < \sqrt{n}, \quad n\in\mathbb{N}, \tag{3.2}$$

and

$$\mathbb{E}[N_n(\mathbb{R})] \le 4\sum_{k=1}^n \frac{[k/2]+1}{k+1}\binom{k}{[k/2]}2^{-k} < 4\sqrt{n}, \quad n\in\mathbb{N}. \tag{3.3}$$

The version for general Müntz polynomials takes the following shape.

Theorem 3.5. *Suppose that $\{c_k\}_{k=0}^\infty$ are i.i.d. real random variables with a common symmetric and absolutely continuous distribution, and $\{\lambda_k\}_{k=1}^\infty$ are arbitrary distinct positive real numbers. Then* (3.2) *holds and*

$$\mathbb{E}[N_n([0,\infty))] \le 2\sum_{k=1}^n \frac{[k/2]+1}{k+1}\binom{k}{[k/2]}2^{-k} < 2\sqrt{n}, \quad n\in\mathbb{N}. \tag{3.4}$$

A result of Siegmund-Schultze and von Weizsäcker on one-dimensional random walks [26] allows us to prove upper bounds on the expected number of positive zeros for polynomials with random coefficients that are not required to have symmetric distribution.

Theorem 3.6. *If $\{c_k\}_{k=0}^\infty$ are arbitrary i.i.d. real random variables and $\{\lambda_k\}_{k=1}^\infty \subset \mathbb{N}$, then*

$$\mathbb{E}[N_n([0,\infty))] = O(\sqrt{n}). \tag{3.5}$$

If $\{c_k\}_{k=0}^\infty$ are i.i.d. real random variables with an absolutely continuous distribution, and $\{\lambda_k\}_{k=1}^\infty$ are arbitrary distinct positive real numbers, then (3.5) *holds true.*

Note that all bounds of the form $O(\sqrt{n})$ given in the results of this section are sharp even for Gaussian coefficients, see Theorem 2.3.

4. Results that hold with probability one

Corollary 3.2 also allows to show that the expected number of real roots is bounded with probability one for biased random coefficients.

Theorem 4.1. *If $\{c_k\}_{k=0}^{\infty}$ are real i.i.d. random variables with $\mathbb{E}[c_0] \neq 0$, and $\{\lambda_k\}_{k=1}^{\infty}$ are arbitrary distinct positive real numbers, then*

$$\limsup_{n\to\infty} \mathbb{E}[N_n([0,1])] < \infty \tag{4.1}$$

holds with probability one.

The proof of Theorem 4.1 relies on the standard Law of Large Numbers, which gives that the sums of coefficients do not change sign for all large $n \in \mathbb{N}$. If we use other versions for the Law of Large Numbers, we can even relax conditions on the coefficients. For example, Theorem 12 in [22, p. 272] gives the following statement, with the same proof as for our Theorem 4.1.

Theorem 4.2. *Let $\{\lambda_k\}_{k=1}^{\infty}$ be arbitrary distinct positive real numbers. Suppose that $\{c_k\}_{k=0}^{\infty}$ are real independent random variables, and there is $\varepsilon > 0$ such that either $\mathbb{E}[c_k] \geq \varepsilon$ for all large k, or $\mathbb{E}[c_k] \leq -\varepsilon$ for all large k. Let $\{a_k\}_{k=0}^{\infty}$ be any sequence of positive numbers strictly increasing to infinity, and such that $a_n = O(n)$. If for a fixed $p \in (1,2]$ we have that*

$$\sum_{k=0}^{\infty} \frac{\mathbb{E}[|c_k - \mathbb{E}[c_k]|^p]}{a_k^p} < \infty, \tag{4.2}$$

then (4.1) *holds with probability one.*

5. Proofs for Section 2

5.1. **Preliminary Estimates.** First we estimate the contribution from zeros in a neighborhood of 0:

Lemma 5.1. *If $\lambda_0 = 0$ and $\lambda_j > 0, j \geq 1$, then for $0 < c < 1$,*

$$\int_0^c I_n(x)\, dx \leq K_n\left(c^{1/2}, c^{1/2}\right).$$

Proof. From (2.1), and as $K_n(x,x) \geq 1$,

$$I_n(x) \leq \sqrt{K_n^{(1,1)}(x,x)} = \sqrt{\sum_{j=1}^{n} \lambda_j^2 x^{2\lambda_j - 2}} \leq \sum_{j=1}^{n} \lambda_j x^{\lambda_j - 1},$$

by the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$. Then

$$\int_0^c I_n(x)\, dx \leq \sum_{j=1}^{n} c^{\lambda_j} \leq K_n\left(c^{1/2}, c^{1/2}\right).$$

□

Next, we establish a compact expression for the numerator in $I_n(x)$. In the sequel, we let

$$\Delta_n(x) = \sum_{j,k=0}^{n} x^{2\lambda_j} x^{2\lambda_k} (\lambda_j - \lambda_k)^2. \tag{5.1}$$

Lemma 5.2.

$$K_n^{(1,1)}(x,x)\,K_n(x,x)-K_n^{(1,0)}(x,x)^2=\frac{\Delta_n(x)}{2x^2}. \tag{5.2}$$

Proof. The left-hand side equals

$$\begin{aligned}
&\frac{1}{x^2}\left(\sum_{j=0}^{n}\lambda_j^2x^{2\lambda_j}\right)\left(\sum_{j=0}^{n}x^{2\lambda_j}\right)-\frac{1}{x^2}\left(\sum_{j=0}^{n}\lambda_jx^{2\lambda_j}\right)^2\\
=\;&\frac{1}{x^2}\left\{\sum_{j=0}^{n}\sum_{k=0}^{n}\left(\lambda_j^2-\lambda_j\lambda_k\right)x^{2\lambda_j}x^{2\lambda_k}\right\}\\
=\;&\frac{1}{x^2}\left\{\sum_{k>j}x^{2\lambda_j}x^{2\lambda_k}\left[\lambda_j^2-2\lambda_j\lambda_k+\lambda_k^2\right]\right\}=\frac{\Delta_n(x)}{2x^2}.
\end{aligned}$$

□

Next, we estimate the sum $\Delta_n(x)$ by an integral. Let

$$J_n(x)=\int_0^n\int_0^n x^{2(\phi(s)+\phi(t))}\left(\phi(s)-\phi(t)\right)^2 ds\ dt. \tag{5.3}$$

For fixed $x\in(0,1)$, let

$$G(s,t)=\log\left[x^{2(\phi(s)+\phi(t))}\left(\phi(s)-\phi(t)\right)^2\right],\ s,t\geq 0, \tag{5.4}$$

and

$$h(t)=\phi^{[-1]}\left(\phi(t)+|\log x|^{-1}\right). \tag{5.5}$$

Here, $\phi^{[-1]}$ is the inverse function of ϕ.

Lemma 5.3. *Fix $0<x<1$ and $t>0$.*
(a) As a function of s, $G(s,t)$ decreases in $(0,t)$, increases in $(t,h(t))$ and decreases in $(h(t),\infty)$.
(b)

$$\max_{s\in[0,t]}e^{G(s,t)}=e^{G(0,t)}=x^{2\phi(t)}\phi(t)^2;$$
$$\max_{s\in[t,\infty]}e^{G(s,t)}=e^{G(h(t),t)}=x^{4\phi(t)}\left(e\,|\log x|\right)^{-2}.$$

(c) $x^{2\phi(t)}\phi(t)^2$ increases as a function of t for $\phi(t)<\frac{1}{|\log x|}$ and decreases thereafter. Moreover

$$\max_{t\geq 0}x^{2\phi(t)}\phi(t)^2=\left(e\,|\log x|\right)^{-2}.$$

Proof. (a) This follows from

$$\frac{\partial G}{\partial s}=2\phi'(s)\left[\log x+\frac{1}{\phi(s)-\phi(t)}\right]$$

for $s\neq t$ as well as the fact that $\phi'>0$ in $(0,\infty)$.
(b) This follows from (a).
(c) We see that

$$\frac{d}{dt}\log\left(x^{2\phi(t)}\phi(t)^2\right)=2\phi'(t)\left[\log x+\frac{1}{\phi(t)}\right],$$

so that the maximum occurs when $\phi(t) = \frac{1}{|\log x|}$. ☐

Lemma 5.4. *Let*

$$E_n(x) = 4(e|\log x|)^{-2}\left[\frac{3}{2} + \int_0^n x^{4\phi(s)} ds\right] + 2\int_0^n x^{2\phi(s)}\phi(s)^2 \, ds.$$

Then

$$|\Delta_n(x) - J_n(x)| \le E_n(x). \tag{5.6}$$

Proof. Fix $t > 0$. Choose an integer r such that $h(t) \in (r, r+1]$. If $h(t) > n$, redefine $r = n$. Then using the previous lemma,

$$\begin{aligned}
\sum_{j=0}^{n} e^{G(j,t)} &= \left(\sum_{j=0}^{r-1} + \sum_{j=r}^{r+1} + \sum_{j=r+2}^{n}\right) e^{G(j,t)} \\
&\le x^{2\phi(t)}\phi(t)^2 + \int_0^r e^{G(s,t)} ds + 2x^{4\phi(t)}(e|\log x|)^{-2} \\
&+ \int_{r+1}^{n} e^{G(s,t)} ds \\
&\le \int_0^n e^{G(s,t)} ds + x^{2\phi(t)}\phi(t)^2 + 2x^{4\phi(t)}(e|\log x|)^{-2}.
\end{aligned}$$

Then applying this with $t = k$,

$$\begin{aligned}
&\Delta_n(x) = \sum_{k=0}^{n}\sum_{j=0}^{n} e^{G(j,k)} \\
\le &\sum_{k=0}^{n}\left(\int_0^n e^{G(s,k)} ds + x^{2\phi(k)}\phi(k)^2 + 2x^{4\phi(k)}(e|\log x|)^{-2}\right) \\
= &\int_0^n \left(\sum_{k=0}^{n} e^{G(s,k)}\right) ds + \sum_{k=0}^{n} x^{2\phi(k)}\phi(k)^2 + 2(e|\log x|)^{-2}\sum_{k=0}^{n} x^{4\phi(k)}
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^n \left(\int_0^n e^{G(s,t)}dt + x^{2\phi(s)}\phi(s)^2 + 2x^{4\phi(s)}\left(e\left|\log x\right|\right)^{-2}\right) ds \\
&\quad + \sum_{k=0}^{n} x^{2\phi(k)}\phi(k)^2 + 2\left(e\left|\log x\right|\right)^{-2}\sum_{k=0}^{n} x^{4\phi(k)} \\
&= J_n(x) + 2\left(e\left|\log x\right|\right)^{-2}\left\{\int_0^n x^{4\phi(s)}ds + \sum_{k=0}^{n} x^{4\phi(k)}\right\} \\
&\quad + \left\{\int_0^n x^{2\phi(s)}\phi(s)^2\, ds + \sum_{k=0}^{n} x^{2\phi(k)}\phi(k)^2\right\} \\
&\le J_n(x) + 4\left(e\left|\log x\right|\right)^{-2}\left[1 + \int_0^n x^{4\phi(s)ds}\right] + 2\left(e\left|\log x\right|\right)^{-2} \\
&+ 2\int_0^n x^{2\phi(t)}\phi(t)^2\, dt \\
&= J_n(x) + E_n(x),
\end{aligned}
$$

by the previous lemma. In a very similar way, we can establish the lower bound

$$\Delta_n(x) \ge J_n(x) - E_n(x).$$

□

We need an alternative form for $J_n(x)$ and the error term $E_n(x)$:

Lemma 5.5. *Let $0 < x < 1$, and define*

$$H(y,x) = \Psi\left(\frac{1}{\left|\log x\right|}\right) / \Psi\left(\frac{y}{\left|\log x\right|}\right). \tag{5.7}$$

We have the following:
(a)

$$
\begin{aligned}
J_n(x) = {} & \left|\log x\right|^{-4}\Psi\left(\frac{1}{\left|\log x\right|}\right)^{-2} \\
& \times \int_0^{\phi(n)\left|\log x\right|}\int_0^{\phi(n)\left|\log x\right|} e^{-2(y+z)}(y-z)^2 H(y,x)H(z,x)\,dy\ dz.
\end{aligned} \tag{5.8}
$$

(b)

$$E_n(x) \le \left|\log x\right|^{-3}\left[\left|\log x\right| + \Psi\left(\frac{1}{\left|\log x\right|}\right)^{-1}\int_0^{\phi(n)\left|\log x\right|}\left[e^{-4y} + 2e^{-2y}y^2\right]H(y,x)\,dy\right]. \tag{5.9}$$

(c)

$$0 \le K_n(x,x) - \left|\log x\right|^{-1}\Psi\left(\frac{1}{\left|\log x\right|}\right)^{-1}\int_0^{\phi(n)\left|\log x\right|} e^{-2y}H(y,x)\,dy \le 1. \tag{5.10}$$

Proof. (a) Make the substitution

$$y = \phi(s)\left|\log x\right| \text{ and } z = \phi(t)\left|\log x\right|$$

in the integral (5.3) defining $J_n(x)$. We obtain

(5.11)

$$\begin{aligned}J_n(x) &= \int_0^n \int_0^n x^{2(\phi(s)+\phi(t))} (\phi(s)-\phi(t))^2\, ds\ dt \\ &= |\log x|^{-4} \int_0^{\phi(n)|\log x|} \int_0^{\phi(n)|\log x|} e^{-2(y+z)} (y-z)^2 \frac{dy}{\Psi\left(\frac{y}{|\log x|}\right)} \frac{dz}{\Psi\left(\frac{z}{|\log x|}\right)} \\ &= |\log x|^{-4}\, \Psi\left(\frac{1}{|\log x|}\right)^{-2} \int_0^{\phi(n)|\log x|} \int_0^{\phi(n)|\log x|} e^{-2(y+z)} (y-z)^2 H(y,x) H(z,x)\, dy\ dz.\end{aligned}$$

(b) Next, the substitution $y = \phi(s)|\log x|$ gives

$$\begin{aligned}E_n(x) &= 4(e|\log x|)^{-2} \left[\frac{3}{2} + \int_0^n x^{4\phi(s)} ds\right] + 2\int_0^n x^{2\phi(s)} \phi(s)^2\, ds \\ &= 4(e|\log x|)^{-2} \left[\frac{3}{2} + |\log x|^{-1} \int_0^{\phi(n)|\log x|} e^{-4y} \frac{dy}{\Psi\left(\frac{y}{|\log x|}\right)}\right] \\ &+ 2|\log x|^{-3} \int_0^{\phi(n)|\log x|} e^{-2y} y^2 \frac{dy}{\Psi\left(\frac{y}{|\log x|}\right)} \\ &\le |\log x|^{-3} \left[|\log x| + \Psi\left(\frac{1}{|\log x|}\right)^{-1} \int_0^{\phi(n)|\log x|} \left[e^{-4y} + 2e^{-2y} y^2\right] H(y,x)\, dy\right].\end{aligned}$$

(c) Here as above,

$$\begin{aligned}K_n(x,x) &\le 1 + \int_0^n x^{2\phi(s)} ds \\ &= 1 + |\log x|^{-1}\, \Psi\left(\frac{1}{|\log x|}\right)^{-1} \int_0^{\phi(n)|\log x|} e^{-2y} H(y,x)\, dy.\end{aligned}$$

Similarly,

$$\begin{aligned}K_n(x,x) &\ge \int_0^n x^{2\phi(s)} ds \\ &= |\log x|^{-1}\, \Psi\left(\frac{1}{|\log x|}\right)^{-1} \int_0^{\phi(n)|\log x|} e^{-2y} H(y,x)\, dy.\end{aligned}$$

Then (5.10) follows. □

5.2. **Proof of Theorem 2.1.** Most of the zeros cluster around 1. Accordingly, we begin with

Lemma 5.6. *Let $\Delta \in \mathbb{R}$. Assume the hypotheses (2.5)-(2.8) of Theorem 2.1. For any sequence $\{x_n\}$ in $(0,1)$ with limit 1, and*

(5.12) $$\lim_{n\to\infty} \phi(n)|\log x_n| = \infty,$$

we have

$$\lim_{n\to\infty} |\log x_n|^2 I_n(x_n)^2 = d_0^2. \tag{5.13}$$

Proof. Recall the definition (5.7) of $H(y,x)$. First note that by Lebesgue's Dominated Convergence Theorem and our hypotheses (2.6)-(2.8),

$$\lim_{n\to\infty} \int_0^{\phi(n)|\log x_n|} e^{-2y} H(y,x_n)\, dy = \int_0^\infty e^{-2y} y^\Delta dy = 2^{-\Delta-1}\Gamma(1+\Delta).$$

Thus from (5.10) (we are implicitly using (2.5)),

$$\lim_{n\to\infty} |\log x_n| \Psi\left(\frac{1}{|\log x_n|}\right) K_n(x_n,x_n) = 2^{-\Delta-1}\Gamma(1+\Delta). \tag{5.14}$$

Also,

$$\begin{aligned}
&\int_0^\infty \int_0^\infty e^{-2(y+z)} (y-z)^2 g(y) g(z)\, dy\, dz \\
\leq\ & 4\left(\int_0^\infty e^{-2y} y^2 g(y)\, dy\right)\left(\int_0^\infty e^{-2z} g(z)\, dz\right) < \infty,
\end{aligned}$$

by our hypothesis (2.8). Again, Lebesgue's Dominated Convergence Theorem gives

$$\begin{aligned}
&\lim_{n\to\infty} \int_0^{\phi(n)|\log x_n|} \int_0^{\phi(n)|\log x_n|} e^{-2(y+z)} (y-z)^2 H(y,x_n) H(z,x_n)\, dy\, dz \\
&= \int_0^\infty \int_0^\infty e^{-2(y+z)} (y-z)^2 y^\Delta z^\Delta dy\, dz \\
&= 2\left(\int_0^\infty e^{-2y} y^{\Delta+2} dy\right)\left(\int_0^\infty e^{-2y} y^\Delta dy\right) - 2\left(\int_0^\infty e^{-2y} y^{\Delta+1} dy\right)^2 \\
&= 2^{-2\Delta-3}\left\{\Gamma(\Delta+3)\Gamma(\Delta+1) - \Gamma(\Delta+2)^2\right\} =: c_0,
\end{aligned} \tag{5.15}$$

say. Then from Lemmas 5.4 and 5.5(a),

$$\begin{aligned}
&|\log x_n|^4 \Psi\left(\frac{1}{|\log x_n|}\right)^2 \Delta_n(x_n) \\
=\ & |\log x_n|^4 \Psi\left(\frac{1}{|\log x_n|}\right)^2 \{J_n(x_n) + O(E(x_n))\} \\
=\ & c_0 + O\left(|\log x_n| \Psi\left(\frac{1}{|\log x_n|}\right)\right) + o(1) \\
=\ & c_0 + o(1),
\end{aligned}$$

by (2.5). Combining this and (5.14) gives

$$\begin{aligned}
\lim_{n\to\infty} |\log x_n|^2 I_n(x_n)^2 &= \lim_{n\to\infty} \frac{|\log x_n|^4 \Psi\left(\frac{1}{|\log x_n|}\right)^2 \Delta_n(x_n)}{|\log x_n|^2 \Psi\left(\frac{1}{|\log x_n|}\right)^2 2x_n^2 K_n(x_n,x_n)} \\
&= \frac{c_0}{2\left(2^{-\Delta-1}\Gamma(1+\Delta)\right)^2} = d_0^2,
\end{aligned}$$

recall (2.10). □

Proof of Theorem 2.1. For a suitable sequence $\{\varepsilon_n\} \subset (0,1)$ with limit 0, and a suitable sequence $\{\zeta_n\}$ with limit ∞, we split

$$\mathbb{E}\left[N_n\left([0,1]\right)\right] = \frac{1}{\pi}\left(\int_0^{1-\varepsilon_n} + \int_{1-\varepsilon_n}^{1-\zeta_n/\phi(n)} + \int_{1-\zeta_n/\phi(n)}^{1}\right) I_n\left(x\right) dx.$$

We shall assume that $\zeta_n \to \infty$ so slowly that

$$\zeta_n = o\left(\log \phi\left(n\right)\right). \tag{5.16}$$

Now we choose the sequence $\{\varepsilon_n\}$. By Lemma 5.1, if $c \in (0,1)$,

$$\int_0^c I_n\left(x\right) dx \leq K_n\left(c^{1/2}, c^{1/2}\right).$$

Here by Lemma 5.5(c), and our hypotheses (2.7)-(2.8), if c is close enough to 1, we have for all $n \geq 1$,

$$K_n\left(c^{1/2}, c^{1/2}\right) \leq \left|\log c^{1/2}\right|^{-1} \Psi\left(\frac{1}{\left|\log c^{1/2}\right|}\right)^{-1} \int_0^\infty e^{-2y} g\left(y\right) dy + 1.$$

It follows that for each $c \in (0,1)$, there is a constant $C_1\left(c\right)$ such that

$$\sup_n \int_0^c I_n\left(x\right) dx \leq C_1\left(c\right).$$

Hence if $\varepsilon_n \to 0$ sufficiently slowly,

$$\int_0^{1-\varepsilon_n} I_n\left(x\right) dx = o\left(\log \phi\left(n\right)\right). \tag{5.17}$$

Next, from Lemma 5.6, and noting that (5.12) holds uniformly in this range of x, and then using the substitution $t = |\log x| \Leftrightarrow x = e^{-t}$,

$$\begin{aligned}
\int_{1-\varepsilon_n}^{1-\zeta_n/\phi(n)} I_n\left(x\right) dx &= \int_{1-\varepsilon_n}^{1-\zeta_n/\phi(n)} d_0 \left|\log x\right|^{-1}\left(1+o\left(1\right)\right) dx \\
&= d_0\left(1+o\left(1\right)\right) \int_{|\log(1-\zeta_n/\phi(n))|}^{|\log(1-\varepsilon_n)|} t^{-1} e^{-t} dt \\
&= d_0\left(1+o\left(1\right)\right) \int_{(\zeta_n/\phi(n))(1+o(1))}^{\varepsilon_n(1+o(1))} t^{-1} dt \\
&= d_0\left(1+o\left(1\right)\right) \log\left(\varepsilon_n \phi\left(n\right)/\zeta_n\right) \\
&= d_0\left(1+o\left(1\right)\right) \log \phi\left(n\right),
\end{aligned} \tag{5.18}$$

if $\varepsilon_n \to 0$ sufficiently slowly and by (5.16). Finally there is the elementary estimate

$$I_n\left(x\right) \leq \frac{\phi\left(n\right)}{\sqrt{2}x},$$

as follows from (5.1) and (5.2), so for large enough n,

$$\int_{1-\zeta_n/\phi(n)}^{1} I_n\left(x\right) dx \leq C\zeta_n = o\left(\log \phi\left(n\right)\right),$$

recall (5.16). Combining this and (5.17)-(5.18), gives (2.9). □

Example 1
Let $\beta > 0$, and

$$\phi(t) = t^{\beta}.$$

Here

$$\Psi(t) = \phi'\left(\phi^{[-1]}(t)\right) = \beta t^{1-1/\beta}.$$

Also H defined by (5.7) satisfies

$$H(y,x) = y^{1/\beta-1}.$$

All the requirements are met and then we can take $g(y) = y^{1/\beta-1}$. Also $\Delta = -1 + \frac{1}{\beta}$. The necessary integrals converge.

Example 2
Let $\beta > 0$, $\gamma > 0$ and

$$\phi(t) = t^{\beta}(\log(t+2))^{\gamma}, t \geq 0.$$

We show that we can still choose $\Delta = -1 + \frac{1}{\beta}$ and find a function g satisfying the conditions of Theorem 2.1. Firstly,

$$\frac{\phi'(t)}{\phi(t)} = \frac{\beta}{t} + \frac{\gamma}{(2+t)\log(2+t)} = \frac{\beta}{t}\left[1 + f(t)\right], \tag{5.19}$$

where

$$f(t) = \frac{\gamma}{\beta}\frac{t}{(2+t)\log(2+t)}, t \geq 0.$$

Note that f is continuous, nonnegative, bounded in $[0,\infty)$ and

$$\lim_{t\to\infty} f(t) = 0.$$

Next, from (5.19),

$$\frac{\Psi(s)}{s} = \frac{\beta}{\phi^{[-1]}(s)}\left[1 + f\left(\phi^{[-1]}(s)\right)\right].$$

This already gives our requirement (2.5). Next,

$$\frac{\Psi(r)}{\Psi(ry)} = \frac{1}{y}\frac{\phi^{[-1]}(ry)}{\phi^{[-1]}(r)}\frac{1 + f\left(\phi^{[-1]}(r)\right)}{1 + f\left(\phi^{[-1]}(ry)\right)}. \tag{5.20}$$

Here

$$\begin{aligned} s^{1/\beta} &= \phi^{[-1]}(s)\left(\log\left(\phi^{[-1]}(s) + 2\right)\right)^{\gamma/\beta} \\ &\Rightarrow \frac{1}{\beta}\log s = \log\phi^{[-1]}(s) + \frac{\gamma}{\beta}\log\log\left(\phi^{[-1]}(s) + 2\right). \end{aligned} \tag{5.21}$$

This gives

$$\lim_{s\to\infty}\frac{\log\phi^{[-1]}(s)}{\log s} = \frac{1}{\beta}.$$

Next from (5.20) and the definition of ϕ,

$$\frac{\Psi(r)}{\Psi(ry)} = y^{-1+\frac{1}{\beta}}\left[\frac{\log\left(\phi^{[-1]}(r) + 2\right)}{\log\left(\phi^{[-1]}(ry) + 2\right)}\right]^{\gamma/\beta}\frac{1 + f\left(\phi^{[-1]}(r)\right)}{1 + f\left(\phi^{[-1]}(ry)\right)}, \tag{5.22}$$

so for fixed $y > 0$,

$$\lim_{r\to\infty} \frac{\Psi(r)}{\Psi(ry)} = y^{-1+\frac{1}{\beta}} \lim_{r\to\infty} \left[\frac{\log r}{\log ry}\right]^{\gamma/\beta} = y^{-1+1/\beta}.$$

Thus we can choose $\Delta = -1 + 1/\beta$. Next, we show there is a dominating function g. We consider three ranges of y :

(I) $y \geq 1$.

Here (5.22), the boundedness and nonnegativity of f, as well as the monotonicity of $\phi^{[-1]}$, give

$$\frac{\Psi(r)}{\Psi(ry)} \leq Cy^{-1+\frac{1}{\beta}}.$$

(II) $y < 1$ and $ry \leq 2$.

Here $r \leq 2/y$, so

$$\begin{aligned} \frac{\Psi(r)}{\Psi(ry)} &\leq Cy^{-1+\frac{1}{\beta}} \left[\frac{\log\left(\phi^{[-1]}(2/y)+2\right)}{\log 2}\right]^{\gamma/\beta} \\ &\leq Cy^{-1+\frac{1}{\beta}} \left(\log\frac{2}{y}\right)^{\gamma/\beta}. \end{aligned} \tag{5.23}$$

(III) $y < 1$ and $ry \geq 2$.

Here since $\frac{\log r}{\log(ry)}$ is a decreasing function of r if $r \geq 2/y$, so

$$\frac{\log\left(\phi^{[-1]}(r)+2\right)}{\log\left(\phi^{[-1]}(ry)+2\right)} \leq C\frac{\log r}{\log(ry)} \leq C\frac{\log(2/y)}{\log 2},$$

and then we again get (5.23). In summary, we have for $y \geq 0$,

$$\frac{\Psi(r)}{\Psi(ry)} \leq g(y) = cy^{-1+1/\beta} \begin{cases} 1, & y \geq 1 \\ \left(\log\frac{2}{y}\right)^{\gamma/\beta}, & 0 < y < 1 \end{cases}.$$

So we have satisfied all the requirements of Theorem 2.1.

5.3. **Proof of Theorem 2.2.** The approach is similar to that of the previous section, but some parts require more care. Throughout this section, we let $\phi(j) = \log(j+1), j \geq 0$, and

$$X = X(x) = 2\log x + 1, x \in (0,1). \tag{5.24}$$

We separately consider x smaller or larger than $e^{-1/2}$.

Lemma 5.7. *Let* $(x_n) \subset \left(0, e^{-1/2}\right)$ *with*

$$\lim_{n\to\infty} X(x_n) = 0 \text{ but } \lim_{n\to\infty} |X(x_n)| \log n = \infty. \tag{5.25}$$

Then

$$I_n(x_n) = e^{1/2} |X(x_n)|^{-1} (1 + o(1)). \tag{5.26}$$

Proof. First observe that for $x \in (0,1)$,

$$K_n(x,x) = \sum_{k=1}^{n+1} k^{2\log x} \leq 1 + \int_1^{n+1} t^{2\log x} dt = \begin{cases} 1 + \frac{(n+1)^X - 1}{X}, & X \neq 0 \\ 1 + \log(n+1), & X = 0 \end{cases}. \tag{5.27}$$

Similarly

$$K_n(x,x) \geq \int_1^{n+2} t^{2\log x} dt = \begin{cases} \frac{(n+2)^X - 1}{X}, & X \neq 0 \\ \log(n+2), & X = 0 \end{cases}. \tag{5.28}$$

Then if $\{x_n\}$ satisfies (5.25), we see that as $n \to \infty$, $n^{X(x_n)} = e^{X(x_n)\log n} \to 0$ and then (5.27), (5.28) give

$$K_n(x_n, x_n) = \frac{1 + o(1)}{|X(x_n)|}. \tag{5.29}$$

Next, recall that (5.6) holds, with J_n given by (5.3). Here the substitution $u = \phi(s) = \log(s+1)$, $v = \phi(t) = \log(t+1)$ gives

$$\begin{aligned} J_n(x) &= \int_0^n \int_0^n x^{2(\phi(s)+\phi(t))} (\phi(s) - \phi(t))^2 \, ds \, dt \\ &= \int_0^{\log(n+1)} \int_0^{\log(n+1)} x^{2(u+v)} (u-v)^2 e^{u+v} \, du \, dv \\ &= |X|^{-4} \int_0^{|X|\log(n+1)} \int_0^{|X|\log(n+1)} e^{-(y+z)} (y-z)^2 \, dy \, dz, \end{aligned} \tag{5.30}$$

where we have made the further substitution $y = |X|\, u$, $z = |X|\, v$, and we are now assuming $x < e^{-1/2}$, so that $X < 0$. If we now assume (5.25), we obtain

$$\begin{aligned} J_n(x_n) &= |X(x_n)|^{-4} \left(\int_0^\infty \int_0^\infty e^{-(y+z)} (y-z)^2 \, dy \, dz + o(1) \right) \\ &= 2|X(x_n)|^{-4} \left(\int_0^\infty e^{-y} y^2 dy \int_0^\infty e^{-y} dy - \left(\int_0^\infty e^{-y} y dy \right)^2 + o(1) \right) \\ &= 2|X(x_n)|^{-4} (1 + o(1)). \end{aligned}$$

The error term from Lemma 5.4 is

$$E_n(x_n) = 4(e|\log x_n|)^{-2} \left[\frac{3}{2} + \int_1^{n+1} t^{2X(x_n)-2} dt \right] + 2 \int_1^{n+2} t^{X(x_n)-1} (\log t)^2 \, dt. \tag{5.31}$$

Since $x_n \to e^{-1/2}$ and $X(x_n) \to 0$, the first term in the last right-hand side is bounded. Next, the substitution $y = |X(x_n)| \log t$ gives

$$\int_1^{n+2} t^{X(x_n)-1} (\log t)^2 \, dt = |X(x_n)|^{-3} \int_0^{|X(x_n)|\log(n+2)} e^{-y} y^2 dy.$$

So $E_n(x_n) = O\left(|X(x_n)|^{-3}\right)$. Then from Lemma 5.4, and (5.29),

$$\Delta_n(x) = 2|X(x_n)|^{-4} (1 + o(1))$$

$$\Rightarrow I_n^2(x_n) = \frac{\Delta_n(x_n)}{2x_n^2 K_n(x_n, x_n)^2} = e|X_n|^{-2} (1 + o(1)).$$

□

Next, we consider $x > e^{-1/2}$:

Lemma 5.8. *Let* $\{x_n\} \subset \left(e^{-1/2}, 1\right)$ *with*

$$\lim_{n\to\infty} X(x_n) \log n = \infty, \tag{5.32}$$

and

$$\lim_{n\to\infty} |\log x_n|^{-1} n^{-X(x_n)} = 0. \tag{5.33}$$

Then

$$I_n(x_n) = (x_n X(x_n))^{-1} (1 + o(1)). \tag{5.34}$$

Proof. Note first that by (5.32),

$$\lim_{n\to\infty} n^{X(x_n)} = \infty.$$

Then (5.27), (5.28) give

$$K_n(x_n, x_n) = \frac{n^{X(x_n)}}{X(x_n)} (1 + o(1)), \tag{5.35}$$

recall $X(x_n) > 0$. Next, as at (5.30), but since $X(x_n) > 0$,

$$J_n(x_n) = X(x_n)^{-4} \int_0^{X(x_n)\log(n+1)} \int_0^{X(x_n)\log(n+1)} e^{y+z} (y-z)^2 \, dy \, dz.$$

We can evaluate this by separating the integrals and integrating by parts. Let $Z_n = X(x_n)\log(n+1)$:

$$\begin{aligned} J_n(x_n) &= 2|X(x_n)|^{-4} \left\{ \left(\int_0^{Z_n} e^y y^2 dy \right) \left(\int_0^{Z_n} e^y dy \right) - \left(\int_0^{Z_n} e^y y dy \right)^2 \right\} \\ &= 2|X(x_n)|^{-4} \left\{ \begin{array}{c} \left(e^{Z_n} Z_n^2 - 2\left(e^{Z_n} Z_n - \left(e^{Z_n} - 1\right)\right)\right)\left(e^{Z_n} - 1\right) \\ - \left(e^{Z_n} Z_n - \left(e^{Z_n} - 1\right)\right)^2 \end{array} \right\} \\ &= 2|X(x_n)|^{-4} \left\{ \begin{array}{c} e^{Z_n} Z_n^2 \left(e^{Z_n} - 1\right) - 2e^{Z_n} Z_n \left(e^{Z_n} - 1\right) + 2\left(e^{Z_n} - 1\right)^2 \\ -e^{2Z_n} Z_n^2 + 2e^{Z_n} Z_n \left(e^{Z_n} - 1\right) - \left(e^{Z_n} - 1\right)^2 \end{array} \right\} \\ &= 2|X(x_n)|^{-4} \left\{ -e^{Z_n} Z_n^2 + \left(e^{Z_n} - 1\right)^2 \right\} \\ &= 2|X(x_n)|^{-4} n^{2X(x_n)} (1 + o(1)). \end{aligned} \tag{5.36}$$

We have to estimate the error term $E_n(x_n)$ differently in different ranges. Recall that

$$E_n(x_n) = 4(e|\log x_n|)^{-2} \left[\frac{3}{2} + \int_1^{n+1} t^{2X(x_n)-2} dt \right] + 2\int_1^{n+2} t^{X_n(x_n)-1} (\log t)^2 \, dt. \tag{5.37}$$

Here except when $|\log x_n|$ is small, the dominant term is (making the substitution $y = X(x_n)\log t$)

$$\begin{aligned} \int_1^{n+2} t^{X_n(x_n)-1} (\log t)^2 \, dt &= X(x_n)^{-3} \int_0^{X(x_n)\log(n+2)} e^y y^2 dy \\ &= O\left(X(x_n)^{-3} [(\log n) X(x_n)]^2 e^{\log n X(x_n)} \right) \\ &= o\left(X(x_n)^{-3} n^{2X(x_n)} \right). \end{aligned} \tag{5.38}$$

(A) $(x_n) \subset \left(e^{-1/2}, e^{-1/5}\right)$.
Here $|\log x_n| \sim 1$, so from Lemma 5.4 and (5.36)-(5.38),

$$\Delta_n (x_n) = 2X (x_n)^{-4} n^{2X(x_n)} (1 + o (1)) . \tag{5.39}$$

(B) $x_n \geq e^{-1/5}$ and (5.33) holds.
Here $2X (x_n) - 2 > -1$, and $X (x_n) \sim 1$, so from (5.32), (5.33) and (5.36)-(5.38),

$$E (x_n) = O \left(|\log x_n|^{-2} \left[1 + n^{2X(x_n)-1}\right]\right) + o \left(n^{2X(x_n)}\right) = o \left(n^{2X(x_n)}\right),$$

so again (5.39) holds. Finally, (5.35), (5.36), and (5.6), give the result. $\square$

Remark
Although we shall not use it in the proof of Theorem 2.2, we note that a simpler calculation shows that

$$\begin{aligned} J_n \left(e^{-1/2}\right) &= \frac{1}{6} (\log (n + 1))^4 ; \\ K_n \left(e^{-1/2}, e^{-1/2}\right) &= (1 + o(1)) \log n; \\ E_n \left(e^{-1/2}\right) &= O (1) , \end{aligned}$$

and hence

$$I_n \left(e^{-1/2}\right) = \sqrt{\frac{e}{12}} (1 + o(1)) \log n.$$

Proof of Theorem 2.2. We split

$$\begin{aligned} &\pi \mathbb{E} [N_n ([0, 1])] \\ = &\left[\begin{array}{c} \int_0^{e^{-1/2}\rho_n} + \int_{e^{-1/2}\rho_n}^{e^{-1/2}\left(1-\frac{\zeta_n}{\log(n+1)}\right)} + \int_{e^{-1/2}\left(1-\frac{\zeta_n}{\log(n+1)}\right)}^{e^{-1/2}\left(1+\frac{\zeta_n}{\log(n+1)}\right)} \\ + \int_{e^{-1/2}\left(1+\frac{\zeta_n}{\log(n+1)}\right)}^{\exp\left(-\frac{\log n}{n}\right)} + \int_{\exp\left(-\frac{\log n}{n}\right)}^{1} \end{array} \right] I_n (x) \, dx \\ = &I^{(1)} + I^{(2)} + I^{(3)} + I^{(4)} + I^{(5)}. \end{aligned}$$

Here $\rho_n \to 1$ sufficiently slowly, while $\zeta_n \to \infty$ and

$$\zeta_n = o (\log \log n) . \tag{5.40}$$

The main contribution comes from $I^{(2)}$ and $I^{(4)}$.
First integral $I^{(1)}$
We observe first that if $c \in \left(0, e^{-1/2}\right)$,

$$\begin{aligned} \int_0^c I_n^2 (x) &\leq \int_0^c K_n^{(1,1)} (x, x) \, dx \\ &= \sum_{j=1}^{n} \frac{\lambda_j^2}{2\lambda_j - 1} c^{2\lambda_j - 1} \\ &\leq \frac{1}{c \log(4/e)} \sum_{j=1}^{\infty} (\log (j + 1))^2 (j + 1)^{X(c)-1} < \infty. \end{aligned}$$

As this bound is independent of n, we can choose $c = \rho_n e^{-1/2}$, with $\rho_n \to 1$ sufficiently slowly so that

$$\int_0^{\rho_n e^{-1/2}} I_n^2(x)\,dx = o(\log\log n)^2 . \tag{5.41}$$

We may also assume that

$$|\log|\log\rho_n|| = o(\log\log n) . \tag{5.42}$$

Then Cauchy-Schwarz shows that

$$I^{(1)} = \int_0^{\rho_n e^{-1/2}} I_n(x)\,dx = o(\log\log n) .$$

Second integral $I^{(2)}$

From Lemma 5.7, and then making the substitution $x = \exp\left(-\frac{1}{2} - \frac{u}{2}\right) \Leftrightarrow u = -(2\log x + 1) = -X(x)$,

$$\begin{aligned} I^{(2)} &= \int_{e^{-1/2}\rho_n}^{e^{-1/2}\left(1-\frac{\zeta_n}{\log(n+1)}\right)} I_n(x)\,dx \\ &= (1+o(1))\,e^{1/2}\int_{e^{-1/2}\rho_n}^{e^{-1/2}\left(1-\frac{\zeta_n}{\log(n+1)}\right)} \frac{1}{|X(x)|}dx \\ &= (1+o(1))\,\frac{1}{2}\int_{\frac{2\zeta_n}{\log n}(1+o(1))}^{2|\log\rho_n|} \frac{1}{u}e^{-u/2}du \\ &= (1+o(1))\,\frac{1}{2}\int_{\frac{2\zeta_n}{\log n}(1+o(1))}^{2|\log\rho_n|} \frac{1}{u}du \\ &= (1+o(1))\,\frac{1}{2}\left\{\log\frac{\log n}{\zeta_n|\log\rho_n|}\right\} \\ &= \frac{1}{2}(1+o(1))\log\log n, \end{aligned}$$

by (5.40) and (5.42).

Third integral $I^{(3)}$

Since $\lambda_j \le \log(n+1)$ for $j \le n$, so $I_n(x) \le \frac{\log(n+1)}{2x}$ and hence

$$\begin{aligned} I^{(3)} &= \int_{e^{-1/2}\left(1-\frac{\zeta_n}{\log(n+1)}\right)}^{e^{-1/2}\left(1+\frac{\zeta_n}{\log(n+1)}\right)} I_n(x)\,dx \\ &\le C\zeta_n = o(\log\log n), \end{aligned}$$

by (5.40).

Fourth Integral $I^{(4)}$

By Lemma 5.8, and then making the substitution $x = \exp\left(-\frac{1}{2} + \frac{u}{2}\right) \Leftrightarrow u = X(x)$,

$$\begin{aligned} I^{(4)} &= \int_{e^{-1/2}\left(1+\frac{\zeta_n}{\log(n+1)}\right)}^{\exp\left(-\frac{\log n}{n}\right)} I_n(x)\,dx \\ &= (1+o(1)) \int_{e^{-1/2}\left(1+\frac{\zeta_n}{\log(n+1)}\right)}^{\exp\left(-\frac{\log n}{n}\right)} \frac{1}{xX(x)}dx \\ &= (1+o(1)) \frac{1}{2} \int_{\frac{2\zeta_n}{\log(n+1)}(1+o(1))}^{1+o(1)} \frac{du}{u} \\ &= (1+o(1)) \frac{1}{2} \log\left(\frac{\log(n+1)}{2\zeta_n}\right) \\ &= (1+o(1)) \frac{1}{2} \log\log n, \end{aligned}$$

by (5.40).

Fifth Integral $I^{(5)}$

Here

$$I^{(5)} \leq C(\log n)(1-\exp\left(-\frac{\log n}{n}\right)) = o(\log\log n).$$

Finally adding the estimates for $I^{(j)}, 1 \leq j \leq 5$, gives

$$\mathbb{E}[N_n([0,1])] = \frac{1}{\pi}(1+o(1))\log\log n.$$

□

5.4. **Proof of Theorem 2.3.** In this section, we let $b > 1$ and

$$\phi(s) = b^s - 1, s \geq 0.$$

Unfortunately the error term $E_n(x)$ in Lemma 5.4 is of the same order as the main term $J_n(x)$ and so we have to proceed differently. We first estimate some tails:

Lemma 5.9. *(a) Let $\{\zeta_n\}$ be a sequence with limit ∞ but with*

$$\zeta_n = o\left(\sqrt{n}\right). \tag{5.43}$$

Then

$$\int_{1-\zeta_n/b^n}^{1} I_n(x)\,dx = o\left(\sqrt{n}\right). \tag{5.44}$$

(b) Let $\{\varepsilon_n\} \subset (0,\infty)$ decrease to 0 but with $n\varepsilon_n \to \infty$. Then

$$\int_0^{\exp\left(-\frac{1}{n\varepsilon_n}\right)} I_n(x)\,dx = o\left(\sqrt{n}\right). \tag{5.45}$$

Proof. (a) This follows from the trivial bound $I_n(x) \leq \lambda_n = b^n - 1$, so that

$$\int_{1-\zeta_n/b^n}^{1} I_n(x)\,dx \leq \zeta_n.$$

(b) Let $e^{-1/2} < c < 1$. Now

$$\int_0^c I_n^2(x)\,dx \le \int_0^c K_n^{(1,1)}(x,x)\,dx \le \frac{C}{c}\sum_{j=1}^n b^j c^{2b^j},$$

where C is independent of n, c. Let $t_0 = -\frac{\log(2|\log c|)}{\log b} > 0$. The function $f(t) = b^t c^{2b^t}$ is increasing in $(0, t_0)$ and decreasing in (t_0, ∞), and moreover,

$$\max_{t>0} f(t) = f(t_0) = \frac{1}{2e|\log c|},$$

so we may continue this as

$$\begin{aligned}\int_0^c I_n^2(x)\,dx &\le C\left[\int_0^\infty b^t c^{2b^t} dt + \frac{1}{e|\log c|}\right]\\ &= C\left[\frac{1}{2(\log b)|\log c|}\int_{2|\log c|}^\infty e^{-u}du + \frac{1}{e|\log c|}\right] \le \frac{C}{|\log c|},\end{aligned}$$

where C is independent of n, c. Choosing $c = \exp\left(-\frac{1}{n\varepsilon_n}\right)$, we obtain via Cauchy-Schwarz,

$$\int_0^c I_n(x)\,dx \le \sqrt{c\frac{C}{|\log c|}} \le C\sqrt{\varepsilon_n n}.$$

□

Next, we obtain asymptotics for K_n and J_n:

Lemma 5.10. *Let $\{x_n\} \subset (0,1)$ be a sequence with*

$$\lim_{n\to\infty} x_n = 1 \quad \text{and} \quad \lim_{n\to\infty} b^n |\log x_n| = \infty. \tag{5.46}$$

(a)

$$K_n(x_n, x_n) = \frac{\log|\log x_n|^{-1}}{\log b} + O(1).$$

(b)

$$J_n(x_n) = \frac{\log|\log x_n|^{-1}}{2(\log b)^2(\log x_n)^2}(1 + o(1)).$$

Proof. (a) Monotonicity and the substitution $u = 2b^t|\log x|$ give

$$K_n(x,x) \le 1 + \frac{1}{x^2}\int_0^\infty x^{2b^t}dt = 1 + \frac{1}{x^2\log b}\int_{2|\log x|}^\infty e^{-u}\frac{du}{u}.$$

An integration by parts shows that

$$\int_{|2\log x_n|}^\infty e^{-u}u^{-1}\,du = x_n^2\log|2\log x_n|^{-1} + \int_0^\infty e^{-u}\log u\,du + o(1).$$

Then

$$K_n(x_n, x_n) \le \frac{\log|\log x_n|^{-1}}{\log b} + O(1).$$

Similarly,

$$
\begin{aligned}
K_n(x,x) &\geq \frac{1}{x^2}\int_0^{n+1} x^{2b^t}dt \\
&= \frac{1}{x^2\log b}\left\{x^2\log|2\log x|^{-1} - \frac{e^{-2b^{n+1}|\log x|}}{2b^{n+1}|\log x|} + \int_{2|\log x|}^{2b^{n+1}|\log x|} e^{-u}\log u\, du\right\},
\end{aligned}
$$

so that

$$
K_n(x_n,x_n) \geq \frac{\log|\log x_n|^{-1}}{\log b} + O(1).
$$

(b) The substitutions $u=b^s, v=b^t$, followed by $y=2|\log x|\,u$, $z=2|\log x|\,v$, give

$$
\begin{aligned}
J_n(x) &= \int_0^n\int_0^n x^{2\left(b^s+b^t-2\right)}\left(b^s-b^t\right)^2 ds\; dt \\
&= \frac{1}{x^4(\log b)^2}\int_1^{b^n}\int_1^{b^n} x^{2(u+v)}(u-v)^2\frac{du}{u}\frac{dv}{v} \\
&= \frac{1}{x^4(\log b)^2(2\log x)^2}\int_{|2\log x|}^{b^n|2\log x|}\int_{|2\log x|}^{b^n|2\log x|} e^{-(y+z)}(y-z)^2\frac{dy}{y}\frac{dz}{z} \\
&= \frac{2}{x^4(\log b)^2(2\log x)^2}\left\{\begin{array}{c}\left(\int_{|2\log x|}^{b^n|2\log x|} e^{-y}y\; dy\right)\left(\int_{|2\log x|}^{b^n|2\log x|} e^{-z}z^{-1}\; dz\right) \\ -\left(\int_{|2\log x|}^{b^n|2\log x|} e^{-y}\; dy\right)^2\end{array}\right\}.
\end{aligned}
$$

As $x=x_n$ satisfies the hypotheses (5.46),

$$
\begin{aligned}
J_n(x_n) &= \frac{2}{x_n^4(\log b)^2(2\log x_n)^2} \\
&\quad\times\left\{\begin{array}{c}\left(\int_0^\infty e^{-y}y\; dy + o(1)\right)\left(\int_{|2\log x_n|}^\infty e^{-z}z^{-1}\; dy + o(1)\right) \\ -(1+o(1))^2\end{array}\right\} \\
&= \frac{2}{x_n^4(\log b)^2(2\log x_n)^2}\left\{\int_{|2\log x_n|}^\infty e^{-z}z^{-1}\; dz - 1 + o(1)\right\}.
\end{aligned} \tag{5.47}
$$

Here an integration by parts shows that

$$
\begin{aligned}
\int_{|2\log x_n|}^\infty e^{-z}z^{-1}\; dz &= x_n^2\log|2\log x_n|^{-1} + \int_0^\infty e^{-z}\log z\; dz + o(1) \\
&= x_n^2\log|\log x_n|^{-1} + O(1).
\end{aligned}
$$

Substituting in (5.47) gives the result. □

Lemma 5.11. *Assume (5.46). Then*

$$
\Delta_n(x_n) \sim \frac{|\log|\log x_n||}{(\log x_n)^2}. \tag{5.48}
$$

Proof. From Lemma 5.4,

$$
\Delta_n(x_n) \leq J_n(x_n) + 4(e|\log x_n|)^{-2}\left[\frac{3}{2} + (1+o(1))\int_0^n x_n^{4b^s}ds\right] + 2x_n^{-2}(1+o(1))\int_0^n x_n^{2b^s}b^{2s}ds.
$$

The substitution $t = 4b^s |\log x_n|$ gives

$$\begin{aligned}\int_0^n x_n^{4b^s} ds &= \frac{1}{\log b} \int_{4|\log x_n|}^{4b^n |\log x_n|} e^{-t} \frac{dt}{t} \\ &\leq \frac{1}{\log b} \left[\int_{4|\log x_n|}^{4} \frac{dt}{t} + \int_4^{\max\{4b^n|\log x_n|,4\}} e^{-t} dt \right] \\ &\leq C \log |\log x_n|^{-1} + C.\end{aligned}$$

Similarly, $t = 2b^s |\log x_n|$ gives

$$\int_0^n x_n^{2b^s} b^{2s} ds = \frac{1}{\log b \left(2 |\log x_n|\right)^2} \int_{2|\log x_n|}^{2b^n |\log x_n|} e^{-t} t dt.$$

Combined with Lemma 5.10, this gives

$$\Delta_n (x_n) \leq C \frac{|\log |\log x_n||}{(\log x_n)^2}. \tag{5.49}$$

For the opposite inequality, we have to go back into the proof of Lemma 5.4. Assume that G is given by (5.4), and $h(t)$ is given by (5.5). Let

$$I_j (t) := \int_j^{j+1} e^{G(j,t)} dt = \int_j^{j+1} x^{2\left(b^j + b^t\right) - 2} \left(b^j - b^t\right)^2 dt.$$

We see that

$$\begin{aligned}I_{j+1} (t) &= b^2 \int_j^{j+1} x^{2b\left(b^j + b^t\right) - 2} \left(b^j - b^t\right)^2 dt \\ &\leq b^2 \int_j^{j+1} x^{2\left(b^j + b^t\right) - 2} \left(b^j - b^t\right)^2 dt = b^2 I_j (t).\end{aligned}$$

It follows that we can omit the 2 terms bracketing the maximum term in our lower bound in Lemma 5.4, while incurring at most a constant factor. Let us make this more precise. Choose an integer r such that $h(t) \in (r, r+1]$. If $h(t) > n$, redefine $r = n$. We have from the above

$$\int_{r-1}^{r} e^{G(s,t)} ds \geq \frac{1}{2} \int_{r-1}^{r} e^{G(s,t)} ds + \frac{1}{2b^2} \int_r^{r+1} e^{G(s,t)} ds.$$

Then as in Lemma 5.4,

$$\begin{aligned}\sum_{j=0}^{n} e^{G(j,t)} &\geq \left[\sum_{j=1}^{r} + \sum_{j=r+1}^{n}\right] e^{G(j,t)} \\ &\geq \int_0^r e^{G(s,t)} ds + \int_{r+1}^{n+1} e^{G(s,t)} ds \\ &\geq \int_0^{r-1} e^{G(s,t)} ds + \frac{1}{2}\int_{r-1}^{r} e^{G(s,t)} ds \\ &\quad + \frac{1}{2b^2}\int_r^{r+1} e^{G(s,t)} ds + \int_{r+1}^{n+1} e^{G(s,t)} ds \\ &\geq \frac{1}{2b^2}\int_0^n e^{G(s,t)} ds.\end{aligned}$$

Iterating this, yields as in the proof of Lemma 5.4,

$$\Delta_n(x) \geq \left(\frac{1}{2b^2}\right)^2 J_n(x).$$

This and Lemma 5.10, give the lower bound corresponding to (5.49). □

Proof of Theorem 2.3. We split

$$\int_0^1 I_n(x)\,dx = \left[\int_0^{\exp\left(-\frac{1}{n\varepsilon_n}\right)} + \int_{\exp\left(-\frac{1}{n\varepsilon_n}\right)}^{1-\zeta_n/b^n} + \int_{1-\zeta_n/b^n}^{1}\right] I_n(x)\,dx. \tag{5.50}$$

The restrictions on (ε_n) and (ζ_n) are as in Lemma 5.9. Here from Lemma 5.9,

$$\int_{1-\zeta_n/b^n}^{1} I_n(x)\,dx + \int_0^{\exp\left(-\frac{1}{n\varepsilon_n}\right)} I_n(x)\,dx = o\left(\sqrt{n}\right). \tag{5.51}$$

Next, from Lemmas 5.10, 5.11, for the range $x \in \left[\exp\left(-\frac{1}{n\varepsilon_n}\right), 1-\zeta_n/b^n\right]$

$$I_n^2(x) \sim \frac{J_n(x)}{K_n(x,x)^2} \sim \frac{\left(\log\left|\log x\right|^{-1}\right)^{-1}}{\left|\log x\right|^2}.$$

Hence

$$\begin{aligned}
&\int_{\exp\left(-\frac{1}{n\varepsilon_n}\right)}^{1-\frac{\zeta_n}{b^n}} I_n(x)\,dx \\
\sim & \int_{\exp\left(-\frac{1}{n\varepsilon_n}\right)}^{1-\frac{\zeta_n}{b^n}} \left(\log|\log x|^{-1}\right)^{-1/2} \frac{dx}{|\log x|} \\
= & \int_{\frac{\zeta_n}{b^n}}^{1-\exp\left(-\frac{1}{n\varepsilon_n}\right)} \left(\log|\log(1-t)|^{-1}\right)^{-1/2} \frac{dt}{|\log(1-t)|} \\
\sim & \int_{\frac{\zeta_n}{b^n}}^{\frac{1}{n\varepsilon_n}(1+o(1))} \left(\log t^{-1}\right)^{-1/2} \frac{dt}{t} \\
= & \int_{\log\frac{1}{n\varepsilon_n}(1+o(1))}^{n\log b-\log\zeta_n} s^{-1/2} ds \\
\sim & \sqrt{n},
\end{aligned}$$

by our hypotheses (5.43), provided $\{\varepsilon_n\}$ decreases sufficiently slowly to 0. This and (5.50), (5.51) give the result. □

5.5. **Proofs for Theorem 2.4 and Corollary 2.5.**

Proof of Theorem 2.4. We first make a crude estimate on the expected number of zeros near the origin, using the same idea as in the proof of Lemma 5.1. Since $K_n(x,x) \ge 1$, we obtain from (2.1)-(2.2) that

$$I_n(x) \le \sqrt{K_n^{(1,1)}(x,x)} = \sqrt{\sum_{k=1}^n \lambda_k^2 x^{2\lambda_k-2}} \le \sum_{k=1}^n \lambda_k x^{\lambda_k-1}.$$

Hence, for any $c \in (0,1)$, we have by (2.13) that

$$\mathbb{E}[N_n([0,c])] = \int_0^c I_n(x)\,dx \le \sum_{k=1}^n c^{\lambda_k} \le C\sum_{k=1}^n c^{(a\log k)/2} = C\sum_{k=1}^n k^{(a\log c)/2}.$$

The latter sums are uniformly bounded for all $n \in \mathbb{N}$ if $a\log c < -2$. Thus we can choose a sufficiently small $s \in (0,1)$ so that $\mathbb{E}[N_n([0,s])]$ is uniformly bounded for all $n \in \mathbb{N}$. To estimate $\mathbb{E}[N_n([s,1])]$, we write the Kac-Rice formula (2.1)-(2.2) in the following equivalent form:

$$\mathbb{E}[N_n([a,b])] = \frac{1}{2\pi}\int_a^b \left(\left(\frac{u_n'(x)}{u_n(x)}\right)' + \frac{u_n'(x)}{xu_n(x)}\right)^{1/2} dx,$$

where $u_n(x) = K_n(x,x) = \sum_{k=0}^n x^{2\lambda_k}$. Then

$$\begin{aligned}
\mathbb{E}[N_n([s,1])] =& \frac{1}{2\pi} \int_s^1 \left(\frac{u_n''(x)}{u_n(x)} + \frac{u_n'(x)}{xu_n(x)} - \left(\frac{u_n'(x)}{u_n(x)} \right)^2 \right)^{1/2} dx \\
\le& \frac{1}{2\pi} \int_s^1 \left(\frac{u_n''(x)}{u_n(x)} + \frac{u_n'(x)}{xu_n(x)} \right)^{1/2} dx = \frac{1}{2\pi} \int_s^1 \left(\frac{u_n'(x)}{u_n(x)} \left(\frac{u_n''(x)}{u_n'(x)} + \frac{1}{x} \right) \right)^{1/2} dx \\
\le& C \left(\int_s^1 \frac{u_n'(x)}{u_n(x)} dx \right)^{1/2} \left(\int_s^1 \left(\frac{u_n''(x)}{u_n'(x)} + \frac{1}{x} \right) dx \right)^{1/2} \quad \text{(by Cauchy-Schwarz)} \\
=& C \left(\log u_n(x) \Big|_s^1 \right)^{1/2} \left(\log u_n'(x) \Big|_s^1 + \log x \Big|_s^1 \right)^{1/2} \le C \sqrt{\log n} \sqrt{\log \sum_{k=1}^n \lambda_k}.
\end{aligned}$$

□

Proof of Corollary 2.5. If we rearrange terms in the random polynomial (1.1), then we obtain a random polynomial with the same distribution function and the same expected number of real zeros because the random coefficients $\{c_k\}_{k=0}^\infty$ are i.i.d. Hence, we can assume that the exponents $\{\lambda_k\}_{k=1}^\infty \subset \mathbb{N}$ form an increasing sequence, so that (2.13) is satisfied. Since $\sum_{k=1}^n \lambda_k = O(n^{p+1})$, we immediately obtain an estimate of the form $\mathbb{E}[N_n([0,1])] = O(\log n)$ from Theorem 2.4.

The next step is to estimate $\mathbb{E}[N_n([1,\infty))]$. Recalling that $\lambda_n = \max_{1\le k\le n} \lambda_k$, we define a new polynomial $Q_n(x) := x^{\lambda_n} P_n(1/x)$, then the roots of P_n on $[1,\infty)$ are reciprocals of the roots of Q_n on $(0,1]$. Thus, we reduce the estimate on $\mathbb{E}[N_n([1,\infty))]$ for P_n to the corresponding estimate on $\mathbb{E}[N_n([0,1])]$ for Q_n. Note that Q_n is a random polynomial with i.i.d. Gaussian coefficients and exponents $\lambda_n - \lambda_k,\ k = 1, 2, \ldots, n$. It is clear that $\sum_{k=1}^n (\lambda_n - \lambda_k) = O(n^{p+1})$, so that $\mathbb{E}[N_n([1,\infty))] = O(\log n)$ follows by applying Theorem 2.4 to Q_n (after rearranging terms).

Finally, we use the change of variable $x \to -x$, and observe that the roots of P_n in $(-\infty, 0]$ are symmetric about the origin to the roots of $Q_n(x) = P_n(-x) = \sum_{k=0}^n (-1)^{\lambda_k} c_k x^{\lambda_k}$ in $[0,\infty)$. Since the random variables $(-1)^{\lambda_k} c_k$ are also i.i.d. Gaussian, we conclude from the first part of this proof that $\mathbb{E}[N_n((-\infty,0])] = O(\log n)$ too. □

5.6. **Proof of Theorem 2.6.** We first prove:

Lemma 5.12. *Let $B \ge e$ and $0 < \rho < 1$. Assume that*

$$\frac{\log B}{B} \le \frac{1}{2} \left| \log \rho \right|. \tag{5.52}$$

Then for $x \in [0, \rho]$,

$$I_n(x)^2 \le \frac{1}{2x^2} \left\{ B^2 + \frac{2 \min\left\{ n x^B, K_n\left(x^{1/2}, x^{1/2}\right) \right\}}{K_n(x,x)} \right\}. \tag{5.53}$$

Proof. We split the sum in $\Delta_n(x)$ into two terms:

$$\begin{aligned}\Delta_n(x) &= \left\{\sum_{j,k:|\lambda_j-\lambda_k|\le B} + \sum_{j,k:|\lambda_j-\lambda_k|>B}\right\} x^{2\lambda_j}x^{2\lambda_k}(\lambda_j-\lambda_k)^2 \\ &= (\Sigma_1+\Sigma_2).\end{aligned}$$

First,

$$\Sigma_1 \le B^2 \sum_{j,k:|\lambda_j-\lambda_k|\le B} x^{2\lambda_j}x^{2\lambda_k} \le B^2 K_n(x,x)^2.$$

Next for a term in Σ_2, we have

$$|\lambda_j-\lambda_k| > B.$$

Suppose for example $\lambda_j > \lambda_k$. Then

$$\lambda_j > \lambda_k + B > B.$$

Note too that $\frac{\log t}{t}$ is decreasing for $t \ge e$, so our hypotheses give

$$\log\rho + \frac{\log\lambda_j}{\lambda_j} \le \frac{1}{2}\log\rho \text{ for } \lambda_j \ge B.$$

Then for $\lambda_j \ge B$, and $x \in [0,\rho]$,

$$\begin{aligned} & x^{2\lambda_j}x^{2\lambda_k}(\lambda_j-\lambda_k)^2 \\ \le\ & x^{2\lambda_j}x^{2\lambda_k}\lambda_j^2 \\ =\ & x^{2\lambda_k}\exp\left(2\lambda_j\left\{\log x + \frac{\log\lambda_j}{\lambda_j}\right\}\right) \\ \le\ & x^{2\lambda_k}\exp\left(2\lambda_j\left\{\log x + \frac{\log B}{B}\right\}\right) \\ \le\ & x^{2\lambda_k}\exp\left(2\lambda_j\left\{\log x + \frac{1}{2}|\log\rho|\right\}\right) \\ \le\ & x^{2\lambda_k}\exp\left(2\lambda_j\left\{\frac{1}{2}\log x\right\}\right) = x^{2\lambda_k}x^{\lambda_j}.\end{aligned}$$

So

$$\begin{aligned}\Sigma_2 &= \sum_{j,k:|\lambda_j-\lambda_k|>B} x^{2\lambda_j}x^{2\lambda_k}(\lambda_j-\lambda_k)^2 \\ &\le \sum_{j,k:\lambda_j>\lambda_k+B} x^{2\lambda_k}x^{\lambda_j} + \sum_{j,k:\lambda_k>\lambda_j+B} x^{2\lambda_j}x^{\lambda_k} \\ &\le 2K_n(x,x)\min\left\{nx^B, K_n\left(x^{1/2},x^{1/2}\right)\right\}.\end{aligned}$$

Then

$$\Delta_n(x) \le \left\{B^2K_n(x,x)^2 + 2K_n(x,x)\min\left\{nx^B, K_n\left(x^{1/2},x^{1/2}\right)\right\}\right\},$$

and then (5.53) follows from (5.2) and (2.1). □

Proof of Theorem 2.6. (a) We choose

$$B = A \log n$$

for some large enough A : we need A so large that

$$\frac{\log (A \log n)}{A \log n} \leq \frac{1}{2} |\log \rho| \text{ for } n \geq 2.$$

Then Lemma 5.12 gives for $x \in [0, \rho]$,

$$I_n(x) \leq \frac{1}{2x^2} \left\{ (A \log n)^2 + 2n\rho^{A \log n} \right\} \leq \frac{1}{x^2} (A \log n)^2,$$

if A is large enough.

(b) Let us consider $t \in [\frac{1}{\rho}, \infty)$. Write $t = x^{-1}$, where $0 < x < \rho$. We have

$$I_n(t) = I_n\left(x^{-1}\right) = \sqrt{\frac{x^2 \Delta_n(x^{-1})}{2K_n(x^{-1}, x^{-1})}},$$

by Lemma 5.2. Let

$$\Lambda = \max\{\lambda_j : 1 \leq j \leq n\};$$
$$\lambda_j^* = \Lambda - \lambda_j \text{ for all } 0 \leq j \leq n.$$

Note that 0 is amongst the new exponents. Then

$$\Delta_n\left(x^{-1}\right) = x^{-4\Lambda} \left\{ \sum_{j,k=0}^{n} x^{2\lambda_j^*} x^{2\lambda_k^*} \left(\lambda_j^* - \lambda_k^*\right)^2 \right\} = x^{-4\Lambda} \Delta_n^*(x),$$

where Δ_n^* corresponds to the exponents $\left\{\lambda_j^*\right\}$. Similarly,

$$K_n\left(x^{-1}, x^{-1}\right)^2 = x^{-2\Lambda} \sum_{j=0}^{n} x^{2\lambda_j^*} = x^{-2\Lambda} K_n^*(x, x),$$

where K_n^* corresponds to the exponents $\left\{\lambda_j^*\right\}$. So

$$I_n\left(x^{-1}\right)^2 = x^2 \frac{\Delta_n^*(x)}{2K_n^*(x, x)^2} = x^4 I_n^*(x)^2.$$

where $I_n^*(x)$ corresponds to the exponents $\left\{\lambda_j^*\right\}$. As above, if $x \in (0, \rho]$,

$$I_n^*(x)^2 \leq \frac{1}{x^2} (A \log n)^2$$

This yields

$$I_n(t)^2 \leq \frac{1}{t^2} (A \log n)^2,$$

if $t \in [\frac{1}{\rho}, \infty)$.

(c) Using (2.1), we obtain that

$$I_n(x)^2 \leq \frac{K_n^{(1,1)}(x, x)}{K_n(x, x)} = \frac{\sum_{k=1}^{n} \lambda_k^2 x^{2\lambda_k - 2}}{\sum_{k=0}^{n} x^{2\lambda_k}} \leq \frac{\max_{1 \leq k \leq n} \lambda_k^2}{x^2} \frac{\sum_{k=1}^{n} x^{2\lambda_k}}{\sum_{k=0}^{n} x^{2\lambda_k}} \leq \frac{\max_{1 \leq k \leq n} \lambda_k^2}{x^2}.$$

□

We present one last elementary tail estimate:

Lemma 5.13. *Let $R > 1$. Then*

$$\int_R^\infty I_n(x)\,dx \le \left(\sum_{j=0}^{n} R^{\lambda_j-\lambda_n}\right)\left(\sum_{k=0}^{n-1} R^{\lambda_k-\lambda_n}\right).$$

Proof. Since $K_n(x,x) \ge x^{2\lambda_n}$,

$$I_n^2(x) \le \frac{1}{2}\sum_{j,k=0}^{n} x^{2(\lambda_j+\lambda_k-2\lambda_n-1)}(\lambda_j-\lambda_k)^2.$$

Using the inequality $\sqrt{a+b} \le \sqrt{a}+\sqrt{b}$ for $a,b \ge 0$, gives

$$I_n(x) \le \frac{1}{\sqrt{2}}\sum_{j,k=0}^{n} x^{\lambda_j+\lambda_k-2\lambda_n-1}|\lambda_j-\lambda_k|.$$

Hence

$$\begin{aligned}\int_R^\infty I_n(x)\,dx &\le \sum_{j,k=0,j\ne k}^{n} R^{\lambda_j+\lambda_k-2\lambda_n}\frac{|(\lambda_j-\lambda_n)+(\lambda_n-\lambda_k)|}{\lambda_n-\lambda_j+\lambda_n-\lambda_k}\\ &\le \sum_{j,k=0,j\ne k}^{n} R^{\lambda_j+\lambda_k-2\lambda_n} \le \left(\sum_{j=0}^{n} R^{\lambda_j-\lambda_n}\right)\left(\sum_{k=0}^{n-1} R^{\lambda_k-\lambda_n}\right).\end{aligned}$$

□

6. Proofs for Section 3

Proof of Theorem 3.4. Since the random coefficients $\{c_k\}_{k=0}^\infty$ are i.i.d., if we rearrange terms in the random polynomial (1.1), then we obtain a random polynomial with the same distribution function and the same expected number of real zeros. Hence, we can always assume that the exponents $\{\lambda_k\}_{k=0}^n$ form an increasing sequence. Note that each $s_k := \sum_{i=0}^k c_i,\ k \in \mathbb{N}$, is a continuous random variable as all $\{c_k\}_{k=0}^\infty$ are continuous, so that $\mathbb{P}(s_k = 0) = 0,\ k \in \mathbb{N}$. Thus, we can estimate $\mathbb{E}[N_n((0,1])]$ by combining Corollary 3.2 and Theorem 3.3:

$$\mathbb{E}[N_n((0,1])] \le \sum_{k=1}^{n} \frac{[k/2]+1}{k+1}\binom{k}{[k/2]}2^{-k}. \tag{6.1}$$

Making the change of variable $x \to -x$, we observe that the roots of P_n in $[-1,0)$ are symmetric about the origin to the roots of $Q_n(x) = P_n(-x) = \sum_{k=0}^n (-1)^{\lambda_k} c_k x^{\lambda_k}$ in $(0,1]$. Since the random variables $(-1)^{\lambda_k}c_k$ are also i.i.d., and have the same symmetric and continuous distribution function as c_k, estimate (6.1) also holds for Q_n, which gives that

$$\mathbb{E}[N_n([-1,0))] = \mathbb{E}[N_n((0,1])] \le \sum_{k=1}^{n} \frac{[k/2]+1}{k+1}\binom{k}{[k/2]}2^{-k}.$$

Using another change of variable $x \to 1/x$, we consider the random polynomials $R_n(x) = x^{\lambda_n}P_n(1/x) = \sum_{k=0}^n c_k x^{\lambda_n-\lambda_k}$, whose roots in $(0,1]$ and $[-1,0)$ match the roots of P_n in $[1,\infty)$ and $(-\infty,-1]$ respectively. It clear that the associated sums of coefficients for R_n are $\tilde{s}_{k,n} := \sum_{i=0}^k c_{n-i},\ k = 0,\ldots,n$, and that the distribution function for $\tilde{s}_{k,n}$ is the same as that of s_k, for all $k = 0,\ldots,n$. Hence the same argument leading to (6.1) applies to R_n, and we obtain the same upper bounds for the expected number of zeros of P_n in $[1,\infty)$ and $(-\infty,-1]$. We summarize all estimates as follows

$$\mathbb{E}[N_n(\mathbb{R})] = \mathbb{E}[N_n((-\infty,1))] + \mathbb{E}[N_n([-1,0))] + \mathbb{E}[N_n((0,1])] + \mathbb{E}[N_n((1,\infty))] \le 4\sum_{k=1}^{n} \frac{[k/2]+1}{k+1}\binom{k}{[k/2]}2^{-k}, \tag{6.2}$$

where we also used that $\mathbb{P}(P_n(x)=0)=\mathbb{P}(c_0=0)=0$ since c_0 has continuous distribution.

To estimate the binomial coefficients in (6.2), we use the approximation of $n!$ by Stirling's asymptotic series. In particular, Robbins [24] provides the following estimates

$$n^n \exp\left(\frac{1}{12n+1}-n\right)\sqrt{2\pi n} < n! < n^n \exp\left(\frac{1}{12n}-n\right)\sqrt{2\pi n}, \quad n\in\mathbb{N}.$$

Applying these estimates in (6.2), we obtain the upper bound $4\sqrt{n}$ stated in (3.3). □

For the case of arbitrary positive real exponents $\{\lambda_k\}_{k=0}^{\infty}$, we approximate them with rational numbers that have the same denominator, and carry over the bounds for the expected number of real zeros from the polynomial case.

Lemma 6.1. *For any sequence of real positive numbers $\{\lambda_k\}_{k=0}^{n}$, let $r_k(j)=m_k(j)/N(j)$ be rational numbers such that $\lim_{j\to\infty} r_k(j)=\lambda_k,\ k=1,\ldots,n$. If $\{c_k\}_{k=0}^{n}$ are i.i.d. random variables with an absolutely continuous distribution, then*

$$\mathbb{E}\left[N_n\left(\sum_{k=0}^{n} c_k x^{\lambda_k},[0,1]\right)\right] \le \limsup_{j\to\infty}\ \mathbb{E}\left[N_n\left(\sum_{k=0}^{n} c_k x^{m_k(j)},[0,1]\right)\right], \quad n\in\mathbb{N}. \tag{6.3}$$

Proof. We first note that the function $f(x)=x^{1/N(j)}$ is a bijection of $[0,1]$ that gives one-to-one correspondence between zeros of $\sum_{k=0}^{n} c_k x^{m_k(j)}$ and $\sum_{k=0}^{n} c_k x^{r_k(j)}$ on $[0,1]$. Hence

$$\mathbb{E}\left[N_n\left(\sum_{k=0}^{n} c_k x^{m_k(j)},[0,1]\right)\right] = \mathbb{E}\left[N_n\left(\sum_{k=0}^{n} c_k x^{r_k(j)},[0,1]\right)\right] \tag{6.4}$$

for all $j\in\mathbb{N}$, as we count zeros without multiplicities. The expectations in (6.3) and (6.4) can be expressed via the following version of Kac-Rice formula found by Zaporozhets [29, Theorem 1] (see also Theorem 2.1 of [9], and let $k=1$ there). We note that the formula below was stated in [29] and [9] for regular polynomials, but it is also valid for Müntz polynomials by the same proof. Let $p(x)$ be the common probability density function for the i.i.d. random variables $\{c_k\}_{k=0}^{n}$. Then we have

$$\mathbb{E}\left[N_n\left(\sum_{k=0}^{n} c_k x^{\lambda_k},[a,b]\right)\right] = \int_a^b\int_{\mathbb{R}^n} p\left(-\sum_{k=1}^{n} c_k x^{\lambda_k}\right)\prod_{k=1}^{n} p(c_k)\left|\sum_{k=1}^{n}\lambda_k c_k x^{\lambda_k-1}\right| dc_1\ldots dc_n\,dx. \tag{6.5}$$

Clearly, analogous formula holds with $r_k(j)$ instead of λ_k:

$$\begin{aligned} \mathbb{E}\left[N_n\left(\sum_{k=0}^n c_k x^{r_k(j)}, [a,b]\right)\right] \\ = \int_a^b \int_{\mathbb{R}^n} p\left(-\sum_{k=1}^n c_k x^{r_k(j)}\right) \prod_{k=1}^n p(c_k) \left|\sum_{k=1}^n r_k(j) c_k x^{r_k(j)-1}\right| dc_1 \dots dc_n \, dx. \end{aligned} \tag{6.6}$$

If $p(x)$ is continuous, then the integrand in (6.6) continuously depends on the exponents $r_k(j),\ k = 1, \dots, n$. Hence the integrand of (6.6) converges to that of (6.5) pointwise as $j \to \infty$, and (6.3) follows from Fatou's Theorem combined with (6.4). If $p(x)$ is an arbitrary density, then we can change it on a set of arbitrarily small measure to make continuous by Lusin's Theorem. This implies convergence of the integrand of (6.6) to that of (6.5) in measure as $j \to \infty$. Thus, we can extract a subsequence j_m, for which this convergence takes place almost everywhere, and Fatou's Theorem applies to this subsequence as before, giving (6.3) in view of (6.4). □

We are ready to deduce Theorem 3.5 from Theorem 3.4.

Proof of Theorem 3.5. If $\{\lambda_k\}_{k=1}^\infty$ are arbitrary distinct positive real numbers, we can approximate them by rational numbers $r_k(j) = m_k(j)/N(j)$ such that $\lim_{j\to\infty} r_k(j) = \lambda_k,\ k = 1, \dots, n$. Applying Lemma 6.1 and Theorem 3.4, we obtain that (3.2) holds for these general exponents $\{\lambda_k\}_{k=1}^\infty$.

As in the proof of Theorem 3.4, we assume that the exponents $\{\lambda_k\}_{k=1}^n$ form an increasing sequence, and make the change of variable $x \to 1/x$. This gives the random polynomials $R_n(x) = x^{\lambda_n} P_n(1/x) = \sum_{k=0}^n c_k x^{\lambda_n - \lambda_k}$, whose roots in $(0, 1]$ correspond to the roots of P_n in $[1,\infty)$. Applying (3.2) to R_n, we obtain the matching estimate for zeros of P_n in $[1,\infty)$, so that

$$\mathbb{E}[N_n([0,\infty)] = \mathbb{E}[N_n((0,1])] + \mathbb{E}[N_n((1,\infty))] \le 2\sum_{k=1}^n \frac{[k/2]+1}{k+1}\binom{k}{[k/2]}2^{-k}.$$

We used above that $\mathbb{P}(P_n(x) = 0) = \mathbb{P}(c_0 = 0) = 0$ as c_0 has absolutely continuous distribution. □

Proof of Theorem 3.6. We first consider the case $\{\lambda_k\}_{k=1}^n \subset \mathbb{N}$, and use some ideas from the proof of Theorem 3.4. If the random variables $\{c_k\}_{k=0}^\infty$ are degenerate, i.e., equal to a constant C with probability one, then the random polynomials (1.1) take the form

$$P_n(x) = C\sum_{k=0}^n x^{\lambda_k}.$$

It is obvious that such polynomials have no positive roots. Assuming now that the random coefficients are non-degenerate, we recall that the probability of $s_n = \sum_{i=0}^n c_i$ to vanish for any given $n \in \mathbb{N}$ does not exceed $C_1/\sqrt{n}$ for a constant $C_1 > 0$ that depends only on the distribution of c_i, see Theorem 1(d) of [25]. It follows from Descartes' rule of signs that $N_n((0,\infty)) \le n$ for any polynomial of the form (1.1). Thus, conditioning on the event $\{s_n = 0\}$, we obtain that

$$\mathbb{E}[N_n((0,\infty))|\{s_n = 0\}] \le C_1\sqrt{n}. \tag{6.7}$$

Next, we condition on the event $\{s_n \neq 0\}$, in which case we can use Corollary 3.2 and estimate $N_n((0,1])$ by $V(s_0,\dots,s_n)$. Theorem 2 of [26] states that the probability of a sign change in the sequence $s_k = \sum_{i=0}^k c_i,\ k \in \mathbb{N}$, at the k-th step, is of the order $k^{-1/2}$, namely $\mathbb{P}(s_{k-1}s_k < 0) = O(k^{-1/2})$. This immediately implies that

$$\mathbb{E}[N_n((0,1])|\{s_n \neq 0\}] \le \mathbb{E}[V(s_0,\dots,s_n)|\{s_n \neq 0\}] \le C_2 \sum_{k=1}^n k^{-1/2} \le C_3\sqrt{n} \tag{6.8}$$

for some positive constants C_2 and C_3. Applying the standard change of variable $x \to 1/x$, we argue as in the proof of Theorem 3.4 to deduce from (6.8) that

$$\mathbb{E}[N_n((0,\infty))|\{s_n \neq 0\}] = O(\sqrt{n}).$$

This bound, combined with (6.7) and the possiblity of one zero at the origin, gives (3.5).

If $\{\lambda_k\}_{k=1}^\infty$ are arbitrary distinct positive real numbers, we repeat the argument already used in the proof of Theorem 3.5, approximating general λ_k by rational numbers, and using Lemma 6.1 together with already proved estimate (3.5) for natural exponents. The change of variable $x \to 1/x$ provides the estimate for $\mathbb{E}[N_n((1,\infty))]$ as before. □

7. Proofs for Section 4

Proof of Theorem 4.1. By the Strong Law of Large Numbers [11, p. 295], we have that

$$\lim_{n\to\infty} \frac{s_n}{n} = \mathbb{E}[c_0] \neq 0 \quad \text{a.s.}$$

Suppose first that $\{\lambda_k\}_{k=1}^n \subset \mathbb{N}$. As before, we assume without loss of generality that the exponents are arranged in the increasing order. If the lim sup in (4.1) is infinite with positive probability, then Corollary 3.2 immediately implies that the number of sign changes in the sequence $s_n = \sum_{k=0}^n c_k$ tends to ∞ as $n \to \infty$ with positive probability. This directly contradicts the above Law of Large Numbers, as s_n must follow the sign of $\mathbb{E}[c_0]$ for sufficiently large values of n with probability one. Hence, (4.1) holds for the case of natural exponents. To prove that it holds for arbitrary distinct positive real exponents, one needs to apply the same approximation argument as in the proof of Theorem 3.5, and use Lemma 6.1. □

Proof of Theorem 4.2. Letting $X_k := c_k - \mathbb{E}[c_k]$, we obtain from Theorem 12 of [22, p. 272] that

$$\frac{\sum_{k=0}^n c_k - \sum_{k=0}^n \mathbb{E}[c_k]}{a_n} = \frac{\sum_{k=0}^n X_k}{a_n} \to 0 \quad \text{a.s.}$$

Suppose that $\mathbb{E}[c_k] \ge \varepsilon$ and $a_n \le Cn$ for all $n \ge N$. Then we have that

$$\begin{aligned}\liminf_{n\to\infty} \frac{s_n}{a_n} &= \liminf_{n\to\infty} \left(\frac{\sum_{k=0}^n c_k - \sum_{k=0}^n \mathbb{E}[c_k]}{a_n} + \frac{\sum_{k=0}^n \mathbb{E}[c_k]}{a_n} \right) \\ &\ge \liminf_{n\to\infty} \frac{n+1}{a_n}\varepsilon \ge C\varepsilon \quad \text{a.s.}\end{aligned}$$

We now follow the proof of Theorem 4.1 to complete the argument. First, we assume that $\{\lambda_k\}_{k=1}^n \subset \mathbb{N}$. If the lim sup in (4.1) is infinite with positive probability, then Corollary 3.2 implies that the number of sign changes in the sequence s_n tends to ∞ as $n \to \infty$ with positive probability. This contradicts the above lim inf estimate, so that (4.1) holds true. Transition to arbitrary real exponents is done again by approximation, as in the proof of Theorem 3.5.

The case $\mathbb{E}[c_k] \le -\varepsilon$ is handled similarly:

$$\limsup_{n\to\infty} \frac{s_n}{a_n} = \limsup_{n\to\infty} \left(\frac{\sum_{k=0}^n c_k - \sum_{k=0}^n \mathbb{E}[c_k]}{a_n} + \frac{\sum_{k=0}^n \mathbb{E}[c_k]}{a_n} \right)$$
$$\le -\varepsilon \limsup_{n\to\infty} \frac{n+1}{a_n} \le -C\varepsilon \quad \text{a.s.}$$

The rest of the proof repeats its first part. □

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